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INFINITE MATRICES WITH ORTHONORMAL ROWS

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Abstract. It is shown that matrices whose rows form an orthonormal set of infinite sequences have some of the significant properties of the finite matrices. Based on this, some rather important instances of the use of these infinite matrices are given.

In what follows every sequence is a sequence of real numbers. For conveniencie, we also refer to a sequence as a *vector* or, more specifically, as a *row* or a *column vector* or simply, as a *row* or *column*.

We are mainly concerned with matrices each row of which is a square sumable infinite sequence, i.e., a sequence $(a_k)_{k\in\omega}$ such that $\sum_{k=1}^{\infty} a_k^2 < \infty$. As usual, we call $\left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2}$ the length of $(a)_{k\in\omega}$. Hence, we may say that we are mainly concerned with matrices each row of which is an *infinite vector of finite length.*

Throughout this paper by *infinite* we always mean *denumerably* infinite.

Remark 1. An advantage of dealing with infinite vectors $\boldsymbol{v}_i = (b_{ik})_{k\in\omega}$ and $\boldsymbol{v}_j = (b_{jk})_{k\in\omega}$ each of finite length is that (as in the case of finite vectors) their inner procuct $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \sum_{k=1}^{\infty} b_{ik}b_{jk}$ always exists and is equal to the product of the length of $\boldsymbol{v}_i$ with the length of the orthogonal projection of $\boldsymbol{v}_j$ on $\boldsymbol{v}_i$. As usual, if $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$ then we say that $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are orthogonal to each other. We observe also that the length of $\boldsymbol{v}_i$ is equal to $(\boldsymbol{v}_i \cdot \boldsymbol{v}_i)^{1/2}$.

It is well known that, without loss of generality, in many cases (using Gram—Smidt porocess) a set of infinite vectors each of finite length can be replaced (as in the case of finite vectors) by an orthonormal set $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \cdot, \boldsymbol{r}_i, \cdot, \boldsymbol{r}_j, \cdot\}$ of infinite vectors each of length 1 (i.e., a *unit vector*) where distinct vectors

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are pairwise orhogonal. Accordingly, in what follows, we consider matrices such as:

$$(1)\qquad \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{i,k} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

whose rows $r_i = (a_{i1}, a_{i2}, a_{i3}, \dots, a_{ik}, \dots)$ for $i = 1, 2, 3, \dots$ form an orthonormal set of vectors, i.e.,

$$(2)\qquad r_i \cdot r_i = \sum_{k=1}^{\infty} a_{ik}^2 = 1 \quad \text{and} \quad r_i \cdot r_j = \sum_{k=1}^{\infty} a_{ik}a_{jk} = 0 \quad \text{for} \quad i \neq j.$$

We call a matrix such as (1), whose rows satisfy (2), a *row orthonormal infinite matrix*. Also, a matrix with an infinite number of rows and columns is called an *infinite square matrix*. Also, a matrix with an infininte number of rows and columns is called an *infintee square matrix*.

Remark 2. From Remark 1 it follows that if r_i is an infinite vector of length 1 and if v is an infinite vector of finite length then (as in the case of finite vectors) the inner product $r_i \cdot v$ is the length of the orthogonal projection of v on r_i. But then an advantage of dealing with an orthonormal set $\{r_1, r_2, r_3, \cdot\}$ of infinite vectors is that (as in the case of finite vectors) we have (Bessel's inequality):

(3) *the sum of the squares of the lengths of the orthogonal projections of v on $r_1, r_2, r_3, \dots$ is less than or equal to the square of the length of v.*

It is not at all obvious that in a row orthonormal infinite (or for that matter even a finite) matrix such as (1) every column must be a vector of length ≤ 1. In other words, it is not at all obvious why (2) must imply $\sum_{i=1}^{\infty} a_{ik}^2 \leq 1$ for every $k = 1, 2, 3, \dots$. This is proved below.

Lemma 1. *Let N be a row orthonormal infinite matrix. Then every column of N is a vector of length ≤ 1*

Proof. Without loss of generality, we prove the conclusion of the Lemma for the first column of N. Let N be given as (1). Thus, based on (2) we must show that

$$(4)\qquad a_{1,1}^2 + a_{2,1}^2 + a_{3,1}^2 + \dots + a_{1,k}^2 + \dots < 1$$

Obviously, we have:

$$(5)\qquad \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots \\ a_{2,1} & a_{2,2} & \cdots \\ \cdots & \cdots & \cdots \\ a_{i,1} & a_{i,2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \cdot \\ 0 \\ \cdot \end{pmatrix} = \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \cdot \\ a_{i,1} \\ \cdot \end{pmatrix}$$

From (5) it follows that:

$$(6)\qquad \begin{array}{rcl} (a_{1,1}, a_{1,2}, \dots) \cdot (1, 0, 0, \dots) &=& a_{1,1} \\ (a_{2,1}, a_{2,2}, \dots) \cdot (1, 0, 0, \dots) &=& a_{2,1} \\ \dots \qquad \dots &=& . \\ (a_{i,1}, a_{i,2}, \dots) \cdot (1, 0, 0, \dots) &=& a_{i,1} \\ \dots \qquad \dots &=& . \end{array}$$

Clearly, (6) shows that $a_{i,1}$ for every $i = 1, 2, 3, \dots$ is the length of the orthogonal projection of vector $(1, 0, 0, \dots)$ on $(a_{i,1}, a_{i,2}, \dots)$. However, since the rows of matrix N form an orthonormal set of vectors and since the length of vector $(1, 0, 0, \dots)$ is 1 we see that (4) follows directly from (3).

Remark 3. Let us observe that the product AB of two matrices A and B is obtained by forming the inner products of the rows of A with the columns of b. Thus, the product of two infinite matrices may not even exist. Clearly, if the rows of an infinite matrix A are of finite length and the columns of a matrix B are infinite vectors also of finite length then, as mentioned earlier; the product AB always exists.

Let N be a row orthonormal infinite square matrix and $N' = I$ where I is the *infinite unit* matrix. On the other hand, from Lemma 1 and Remark 3 it follows that $N'N$ exists, however, $NN \neq I$ in general. This can be seen from the following example.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \dots & & \dots & & & \dots \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & & \dots & & & \dots \end{pmatrix}$$

Clearly, $AA' = I$ whereas $A'A \neq$ I since the first row of $A'A$ is a zero infinite row vector.

Thus, if N is a row otrhonormal infinite square matrix, we have:

$$(7)\qquad NN' = I \quad \textit{whereas in general,} \quad NN \neq I.$$

Let us observe that if A, B, C are infinite matrices and if $(AB)C$ exists it need not be (associative) equal to $A(BC)$. However, if N is a row orthonormal infinite square matrix and C an infinite column vector of finite length then $(NN')C = C = N(N'C)$. The first equality is not surprizing since it follows readily from (7). However, the second equality must be proved as done below. Also, from (3) it follows readily that $(NC) \cdot (NC) \leq C \cdot C$. On the other hand, as shown in Lemma 2, surprizingly enough $(N'C) \cdot (N'C) = C \cdot C$.

Remark 4. The infinite distributivity of the inner product with respect to addition does not hold in general. Thus, if A is a vector and $(B_i)_{i \in \omega}$ a set of vectors such that $A \cdot \sum_{i=1}^{\infty} B_i$ as well as $\sum_{i=1}^{\infty} A \cdot B_i$ is meaningful then $A \cdot \sum_{i=1}^{\infty} B_i$ need not be equal to $\sum_{i=1}^{\infty} A \cdot B_i \left(\text{unless } \sum_{i=1}^{\infty} B_i \text{ converges in norm} \right)$. However, if $\{N_1, N_2, N_3, \ldots\}$ is an orthonormal set of vectors and $(a_i)_{i \in \omega}$ and $(b_i)_{i \in \omega}$ are vectors of finite length, then obviously the following infinite distributivity of the inner product with respect to addition holds:

$$(8) \qquad \left(\sum_{i=1}^{\infty} N_i a_i \right) \cdot \left(\sum_{i=1}^{\infty} N_i b_i \right) = \sum_{i=1}^{\infty} a_i b_i \, .$$

Naturally, prior to stating Lemma 2, we must observe that, in view of Remark 3 and Lemma 1, the product $N'C$ of the infinite square matrix N' and the infinite column vector C of finite length exist.

As ussual, A' always denotes the transpose of A.

Now, we prove:

Lemma 2. *Let N be a row orthonormal infinite square matrix and C be an infinte column vector of finite length. Then*

$$(9) \qquad (N'C) \cdot (N'C) = C \cdot C \, .$$

Proof. Let $N_1, N_2, N_3, \ldots$ be the rows of N and let $c_1, c_2, c_3, \ldots$ be the coordinates of C. Clearly (and this is a subtle point),

$$(10) \qquad N'C = N_1' c_1 + N_2' c_2 + N_3' c_3 + \ldots$$

Since $\{N_1, N_2, N_3, \ldots\}$ is an orthonormal set of (row) vectors it follows that $\{N_1', N_2', N_3', \ldots\}$ is also an orthonormal set of (column) vectors. Therefore, from (8) it follows that:

$$(11) (N_1' c_1 + N_2' c_2 + N_3' c_3 + \ldots) \cdot (N_1' c_1 + N_2' c_2 + N_3' c_3 + \ldots) = c_1^2 + c_2^2 + c_3^2 + \ldots$$

But then (9) follows from (10) and (11) since $c_1, c_2, c_3, \ldots$ are the coordinates of vector C.

Obiously, from (9) it follows that $N'C$ is a vector of finite length. Thus, the product $N(N'C)$ exists, and, as mentioned above, we prove:

Lemma 3. *Let N be a row orthonormal infinite square matrix and C be an infinite column vector of finite length. Then*

$$(12) \qquad N(N'C) = C \, .$$

Proof. Again, let $N_1, N_2, N_3, \ldots$ be the rows of N and let $c_1, c_2, c_3, \ldots$ be the coordinates of C. From (10) it follows readily that the i-th coordinate of $N(N'C)$ is equal to $N_i' \cdot (N_1' c_1 + N_2' c_2 + N_3' c_3 + \ldots)$. However, again since $\{N_1', N_2', N_3', \ldots\}$ is an orthonormal set of vectors, by (8) we have

$$N_i' \cdot (N_1'c_1 + N_2'c_2 + N_3'c_3 + \ldots) = c_1 \quad \textit{for} \quad i = 1, 2, 3, \ldots,$$

which implies the validity of (12).

From (7) and (12) it follows that if N is a row orthonormal infinite square matrix and C an infinite column vector (or, for that matter, an infinite square matrix whose columns are) of finite length, then the following distributivity holds:

$$(NN')C = N(N'C) = C. \tag{13}$$

From (13) we see that if N is a row orthonormal infinite square matrix then its transpore N' acts almost like a right inverse of N. However, because of the second inequality in (7) we see that N' cannot be called a right inverse of N.

Remark 5. Examining our proofs of Lemmas 2 and 3 we can readily see that in them N need not be necessarily an infinite square matrix. Indeed the same proofs show that the conclusions of Lemmas 2 and 3 hold if N is any (finite or infinite) row orthonormal matrix and C is any (finite or infinite column vector of finite length such that N and C have the same number of rows. Thus we have:

Corollary 1. *Let N be a row orthonormal (finite or infinite) matrix and C be a (finite or infinite column vector of finite length such that N and C have the same number of rows, then*

$$(NN')C = N(N'C) = C \quad \textit{and} \quad (N'C) \cdot (N'C) = C \cdot C. \tag{14}$$

Corollary 1 has numerous applications in solving finite or infinite systems of finite or infinite linear equations.

An example of the application of Corollary 1 is given below.

Theorem 1. *Let*

$$\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \ldots = c_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \ldots = c_2 \\ a_{3,1}x_1 + a_{3,2}x_3 + a_{3,3}x_3 + \ldots = c_3 \\ \ldots\ldots\ldots\ldots\ldots \qquad = \ldots \end{array} \tag{15}$$

be a (finite or infinite) system of finite or infinite linear equations where matrix N of coefficients a_{ij} is row orthonormal and constants c_1, c_2, c_3, ... form a column vector C of finite length such that N and C have the same number of rows. Then

$$\begin{array}{l} x_1 = a_{1,1}c_1 + a_{2,1}c_2 + a_{3,1}c_3 + \ldots = m_1 \\ x_2 = a_{1,2}c_1 + a_{2,2}c_2 + a_{3,2}c_3 + \ldots = m_2 \\ x_3 = a_{1,3}c_1 + a_{2,3}c_2 + a_{3,3}c_3 + \ldots = m_3 \\ \ldots = \quad \ldots\ldots\ldots\ldots\ldots \qquad = \ldots \end{array} \tag{16}$$

is a solution of system (15). *Moreover, this solution* $(x_i = m_i)_{i \in \omega}$ is such that:

$$m_1^2 + m_2^2 + m_3^2 + \ldots = c_1^2 + c_2^2 + c_3^2 + \ldots \tag{17}$$

Proof. Obviously, in matrix notation (15) is written as $NX = C$ and (16) as $X = N'C$. But then from (14) it follows that $N(N'C) = C$ implying that $X = N'C$ is a solution of $NX = C$. Thus, indeed (16) is a solution of system (15). On the other hand, (17) follows directly from the second equality in (14).

Remark 6. Let us observe that the hypotheses of Theorem 1 ensure the existence of a solution of system (15). In fact, (16) provides an explicit solution of system (15). Obviously, in general, nothing prevents system (15) from having infinitely many solutions (as shown in the sequel). However, we prove below that remarkable fact, that, under the hypotheses of Theorem 1, system (15) has a unique solutieon $(x_i = m_i)_{i\in\omega}$ which satisfies (17).

Theorem 2. *Let* (15) *by the system of linear equations described in Theorem* 1. *Let* $(x_i = s_i)_{i\in\omega}$ *be any solution of* (15). *Then*

$$\sum_{i=1}^{\infty} c_i^2 \leq \sum_{i=1}^{\infty} s_i^2. \tag{18}$$

On the other hand, solution $(x_i = m_i)_{i\in\omega}$ given by (16) is the unique solution of system (15) such that:

$$\sum_{i=1}^{\infty} m_i^2 = \sum_{i=1}^{\infty} c_i^2. \tag{19}$$

Proof. Let $\{N_1, N_2, N_3, \dots\}$ be the orthonormal set of the rows of matrix N of coefficients a_{ij} of (15) and let $(x_i = s_i)_{i\in\omega}$ be any solution of (15). Clearly, (15) shows that the length of the orthogonal projection of $(s_1, s_2, s_3, \dots)$ on N_i is equal to c_i for every $i = 1, 2, 3, \dots$ But then, (18) follows readily from (3).

Now, in addition to solution $(x_i = m_i)_{i\in\omega}$(which satisfies (19)) of system (15), let $(x_i = p_i)_{i\in\omega}$ be a solution of (15) such that

$$\sum_{i=1}^{\infty} p_i^2 = \sum_{i=1}^{\infty} c_i^2. \tag{20}$$

To prove that system (15) has a unique solution satisfying (19), we show that $m_i = p_i$ for every $i = 1, 2, 3, \dots$. Since $(x_i = m_i)_{i\in\omega}$ as well as $(x_i = p_i)_{i\in\omega}$ is a solution of system (15) so is $\left(x_i = \frac{1}{2}(m_i + p_i)\right)_{i\in\omega}$. Thus (18) we have

$$\sum_{i=1}^{\infty} c_i^2 \leq \sum_{i=1}^{\infty} \left(\frac{1}{2}(m_i + p_i)\right)^2. \tag{21}$$

On the other hand, by the well known triangle inequality and (19) and (20) we have:

$$\sum_{i=1}^{\infty} \left(\frac{1}{2}(m_i + p_i)\right)^2 \leq \left(\left(\sum_{i=1}^{\infty}\left(\frac{1}{2}m_i\right)^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty}\left(\frac{1}{2}p_i\right)^2\right)^{\frac{1}{2}}\right)^2 = \sum_{i=1}^{\infty} c_i^2 \tag{22}$$

which, in view of (21) implies that in the triangle inequality (22), the inequality sign can be replaced by the equality sign. But (it is well known) that the latter is possible if and only if vectors $(m_i)_{i\in\omega}$ and $(p_i)_{i\in\omega}$ are coordinatewise equally proportional, i.e., there exists a real number k such that $m_i = kp_i$ for every $i = 1$, 2, 3, ... But this, by (19) and (20) implies that $m_i = p_i$ for every $i = 1, 2, 3, \ldots$, as desired.

Remark 7. Theorem 2 shows that if condition (19) is imposed on a solution $(x_i = m_i)_{i\in\omega}$ of system (15) of linear equations described in Theorem 1, then system (15) has a unique such solution. We give below another condition on the matrix of coefficients a_{ij} of system (15) which also implies the (existence and) uniqueness of solution of (15).

Let us recall that an orthonormal set $\{\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3, \ldots\}$ of infinite (or for that matter finite) vectors is called *complete* if and only if the zero infinite vector (0) is the only vector of finite length which is ortogonal to every $\boldsymbol{K}_i$, i.e., for every $i = 1, 2, 3, \ldots$,

(23) $$\boldsymbol{K}_i \cdot \boldsymbol{V} = 0 \quad \textit{if and only if} \quad \boldsymbol{V} = (0).$$

It is well known that any complete orthonormal set of (infinite) vectors is denumerable and that every orthonormal set of vectors can be enlarged (say, by virtue of Zorn's lemma) to a complete orthonormal set.

Remark 8. Let us observe that if $\{\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3, \ldots\}$ is a complete orthonormal set of vectors then with respect to this set the inequality mentioned in (3) becomes an equality. Thus, if $\boldsymbol{v}$ is any vector of finite length, we first of all have:

(24) *$\boldsymbol{v}$ is equal to the sum of its orthogonal projections on $\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3, \ldots$*

Moreover we have (Bessel's, equality):

(25) *the sum of the squares of the lengths of the orthogonal projections of $\boldsymbol{v}$ on $\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3, \ldots$ is equal to the square of the length of $\boldsymbol{v}$.*

To prove (24), we proceed as follows. Since $(\boldsymbol{K}_i)_{i\in\omega}$ is an orthonormal set of vectors, the sum $\sum_{i=1}^{\infty} (\boldsymbol{K}_i \cdot \boldsymbol{v})\boldsymbol{K}_i$ of the projections of $\boldsymbol{v}$ on $(\boldsymbol{K}_i)_{i\in\omega}$ (3) is less than or equal to the length of $\boldsymbol{v}$. Thus, $\sum_{i=1}^{\infty} (\boldsymbol{K}_i \cdot \boldsymbol{v})\boldsymbol{K}_i$ is a vector of finite length. Clearly, $\boldsymbol{v} - \sum_{i=1}^{\infty} (\boldsymbol{K}_i \cdot \boldsymbol{v})\boldsymbol{K}_i$ is a vector orthogonal to every $\boldsymbol{K}_i$ and since $(\boldsymbol{K}_i)_{i\in\omega}$ is complete, by (23) we must have:

(26) $$\boldsymbol{v} = \sum_{i=1}^{\infty} (\boldsymbol{K}_i \cdot \boldsymbol{v})\boldsymbol{K}_i$$

which establishes (24). But then (25) follows from (26) by forming the inner product of each side of the equality sign in (26) with itself.

As expected, we call a matrix whose rows form a complete orthonormal set a *row complete orthonormal matrix*. All such (finite or infinite) matrices are square matrices.

Theorem 3. *Let* $\boldsymbol{K}$ *be a row complete orthonormal (finite or infinite) matrix. Then the transpose* $\boldsymbol{K}'$ *of* $\boldsymbol{K}$ *is also a row complete orthonormal matrix.*

Proof. Let $\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3, \ldots$ represent the rows of $\boldsymbol{K}$ and $\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \ldots$ those of $\boldsymbol{K}'$. First we show that $\{\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3, \ldots\}$ forms an orthonormal set. To this end, without loss of generality, it is enough to show that

(27) $$\boldsymbol{T}_1 \cdot \boldsymbol{T}_1 = 1 \quad \text{and} \quad \boldsymbol{T}_1 \cdot \boldsymbol{T}_2 = 0.$$

Let

(28) $$\boldsymbol{E}_1 = (1, 0, 0, 0, \ldots) \quad \text{and} \quad \boldsymbol{E}_2 = (0, 1, 0, 0, \ldots).$$

As in the case of (6), the first row $\boldsymbol{T}_1$ of $\boldsymbol{K}'$ is given by:

(29) $$\boldsymbol{T}_1 = (\boldsymbol{K}_1 \cdot \boldsymbol{E}_1, \boldsymbol{K}_2 \cdot \boldsymbol{E}_1, \boldsymbol{K}_3 \cdot \boldsymbol{E}_1, \ldots)$$

However, since $\{\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3, \ldots\}$ is a complete orthonormal set of vectors, by (25) we have that the square of the length of $\boldsymbol{T}_1$ is equal to the square of the length of $\boldsymbol{E}_1$ which establishes the first equality in (27). Again since $\{\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3, \ldots\}$ is a complete orthonormal set, by (24) we have:

(30) $$\boldsymbol{E}_1 = (\boldsymbol{K}_1 \cdot \boldsymbol{E}_1)\boldsymbol{K}_1 + (\boldsymbol{K}_2 \cdot \boldsymbol{E}_1)\boldsymbol{K}_2 + (\boldsymbol{K}_3 \cdot \boldsymbol{E}_1)\boldsymbol{K}_3 + \ldots$$

As in the case of (29), the second row $\boldsymbol{T}_2$ of $\boldsymbol{K}'$ is given by:

$$\boldsymbol{T}_2 = (\boldsymbol{K}_1 \cdot \boldsymbol{E}_2, \boldsymbol{K}_2 \cdot \boldsymbol{E}_2, \boldsymbol{K}_3 \cdot \boldsymbol{E}_2, \ldots)$$

Thus, by (29) we have:

(31) $$\boldsymbol{T}_1 \cdot \boldsymbol{T}_2 = \boldsymbol{K}_1 \cdot \boldsymbol{E}_1)(\boldsymbol{K}_1 \cdot \boldsymbol{E}_2) + (\boldsymbol{K}_2 \cdot \boldsymbol{E}_1)(\boldsymbol{K}_2 \cdot \boldsymbol{E}_2) + (\boldsymbol{K}_3 \cdot \boldsymbol{E}_1)(\boldsymbol{K}_3 \cdot \boldsymbol{E}_2) + \ldots$$

On the other hand, from (30) and (28) it follows that

$$\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 = 0 = (\boldsymbol{K}_1 \cdot \boldsymbol{E}_1)(\boldsymbol{K}_1 \cdot \boldsymbol{E}_2) + (\boldsymbol{K}_2 \cdot \boldsymbol{E}_1)(\boldsymbol{K}_2 \cdot \boldsymbol{E}_2) + (\boldsymbol{K}_3 \cdot \boldsymbol{E}_1)(\boldsymbol{K}_3 \cdot \boldsymbol{E}_2) + \ldots$$

which, in view of (31) establishes the second equality in (27).

It remains to show that $\boldsymbol{K}'$ is a row complete matrix. To this end, we show that for every vector $\boldsymbol{V}$ ov finite length we have:

(32) $$\boldsymbol{K}'\boldsymbol{V} = 0 \quad \text{implies} \quad \boldsymbol{V} = (0)$$

which in turn would imply that (0) is the only vector of finite length which is perpendicular to every row of $\boldsymbol{K}'$, which by (23) would assert that $\boldsymbol{K}'$ is row complete.

Now, let $\boldsymbol{K}'(\boldsymbol{V}) = (0)$. But then

$$(33) \qquad \boldsymbol{K}(\boldsymbol{K}'\boldsymbol{V}) = \boldsymbol{K}(0) = 0$$

which by (12) establishes (32).

Theorem 3 can be rephrased as follows:

Corollary 2. *The rows of a (finite or infinite) matrix form a complete orthonormal set of vectors if and only if its columns form a complete orthonormal set of vectors.*

We observe that (32) was established due to the fact that $\boldsymbol{K}'$ turned out to be row orthonormal. Hence, we also have:

Theorem 4. *A (finite or infinite) matrix is both row and column complete orthonormal if and only if it is both row and column orthonormal.*

Remark 9. As indicated in (7), if $\boldsymbol{N}$ is a rov orthonormal square matrix, then $\boldsymbol{N}'\boldsymbol{N} \neq \boldsymbol{I}$, in general. However, if $\boldsymbol{K}$ is a row complete orthonormal matrix, then since $\boldsymbol{K}'$ is also row othonormal, from the equality in (7) if follows that $\boldsymbol{K}'\boldsymbol{K} = \boldsymbol{I}$ since $\boldsymbol{K}'' = \boldsymbol{K}$. Consequently, the transpose $\boldsymbol{K}'$ of a row complete orthonormal matrix $\boldsymbol{K}$ is the inverse of $\boldsymbol{K}$ in the usual sense. Moreover, by virtue of (14), we have:

Theorem 5. *Let $\boldsymbol{K}$ be a row complete orthonormal (finite or infinite) matrix. Then the transpose $\boldsymbol{K}'$ of $\boldsymbol{K}$ is the inverse of $\boldsymbol{K}$ in the usual sense, i.e.,*

$$(34) \qquad \boldsymbol{K}\boldsymbol{K}' = \boldsymbol{K}'\boldsymbol{K} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{K}(\boldsymbol{K}'\boldsymbol{M}) = \boldsymbol{K}'(\boldsymbol{K}\boldsymbol{M}) = \boldsymbol{M}$$

for any matrix $\boldsymbol{M}$ whose columns are vectors of finite length and such that $\boldsymbol{K}$ and $\boldsymbol{M}$ have the same number of columns.

Remark 10. As mentioned in Remark 7, system (15) of linear equations described in Theorem 1, has infinitely many solutions, in general. For instance,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ & & \dots & & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \dots \end{pmatrix} = \begin{pmatrix} c_1, \\ c_2, \\ c_3 \\ c_4 \\ \dots \end{pmatrix}$$

has infinitely many solutions given by $x_1 = a, x_2 = c_1, x_3 = c_2, x_4 = c_3, \dots$, where a is an arbitrary real number.

In contrast to the above, and in view of (34), we have:

Theorem 6. *System (15) of linear equations described in Theorem 1 has a unique solution if and only if the matrix of coefficients a_{ij} is a row complete orthonormal matrix. Otherwise, it has infinitely many solutions.*

Proof. Let the matrix of coefficients a_{ij} be a row complete orthonormal matrix $\boldsymbol{K}$. In matrix notation (15) is rewritten as $\boldsymbol{K}\boldsymbol{X} = \boldsymbol{C}$. Since $\boldsymbol{K}$ is row complete orthonormal, by (34) we have $K'(KX) = X = K'C$, implying the uniqueness of the solution.

To complete the proof of the Theorem, we must show that under the hypothesis that the matrix (a_{ij}) of the coefficients of system (15) is orthonormal but not complete the system (15) has infinitely many solutions. Indeed, in view of the hypothesis, we can always precede system (15) with one extra equation so that the coefficient matrix of the resulting system

$$(35)\qquad \begin{array}{l} b_{1,1}x_1 + b_{1,2}x_2 + b_{1,3}x_3 + b_{1,4}x_4 + \ldots = k_1 \\ a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 + \cdots = c_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + a_{2,4}x_4 + \ldots = c_2 \\ \ldots \qquad\qquad\qquad\qquad\qquad\qquad\qquad = \ldots \end{array}$$

be stil orthonormal and $k_1^2 + \sum_{i=1}^{\infty} c_i^2 < \infty$. But then since k_1 can be chosen arbitrarily, from Theorem 1 it follows that system (35) has infinitely many solutions.

Remark 11. We may sumarize the results of this paper as follows. Let $NX = C$ be a (finite or infinite) system of (finite or infinite) linear equations written in matrix notation. We always assume that N is row orthonormal and that C is a column vector of finite length. Then, the system has a unique solution if and only if N is row complete orthonormal. Otherwise, the system has infinitely many solutions, however, it has a unique solution whose length is equal to the length of C.

Although we always considered systems $NX = C$ where N is a row orthonormal matrix, our results apply to any system $MX = C$ the rows of whose coeficient matrix M can be orthonormalized by the well known Gram—Schmidt process.

For general reference see below.

REFERENCE

1. Cooke, R. G.: Infinite Matricces and sequence Spaces, Dover Pub. Co., New York, 1955.

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SÚHRN
NEKONEČNÉ MATICE S ORTONORMÁLNYMI RIADKAMI

Alexander Abian, Iowa

V práci je ukázané, že matice, ktorých riadky tvoria ortogormálny systém nekonečných postupností, majú niektoré významné vlastnosti konečných matíc. Na základe toho sú dokázané niektoré zaujímavé vlastnosti nekonečných matíc.

РЕЗЮМЕ
БЕСКОНЕЧНЫЕ МАТРИЦЫ С ОРТОНОРМАЛЬНЫМИ СТРОКАМИ

Александер Абиан, Йова

Показано, что матрицы, строки которых образуют ортонормальное множество бесконечных последновательностей, обладают некоторыми важными свойствами конечных матриц. Основываясь на этом, доказаны некоторые интересные свойства бесконечных матриц.