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ELEMENTARY NONSTANDARD APPROACH TO TOPOLOGICAL SPACES

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Introduction

An analogy of the intuitive notion of an infinitesimal quantity is the intuitive relation of an infinitesimal nearness in topological spaces. "x is infinitesimaly near to y" means "x lies in every neighbourhood of the element y" in our conception. Analogically to an application of an infinitesimal quantity in the differential calculus, the application of the relation of an infinitesimal nearness gives a new and objective view on some topological notions.

Methods of the nonstandard analysis make possible to introduce and to apply the relation of an infinitesimal nearness on suitably enriched topological spaces. It is done by applying a relatively difficult apparatus of the mathematical logic. Some interesting results are obtainable by means of a less exacting process which utilises the so-called ultrapower construction. In applying the mentioned construction of the set completion the choice of the concrete index set has an important role. This paper shows one possible choice of the index set in the case of topological spaces and its application in proving nonstandard descriptions of some topological notions. In part IV posibilities of applications of nonstandard analsis in proving some assertions in topology are documented.

I. Construction of nonstandard elements

Definition 1. Let I be a given non-empty set. A system $\mathscr{F}$ of subsets of set I is called an ultrafilter on I iff

(i) $\emptyset \notin \mathscr{F}$, $I \in \mathscr{F}$,

(ii) $A \in \mathscr{F} \wedge B \in \mathscr{F} \Rightarrow A \cap B \in \mathscr{F}$,

(iii) $A \in \mathscr{F} \wedge A \subset B \subset I \Rightarrow B \in \mathscr{F}$,

(iv) either $A \in \mathscr{F}$ or $I - A \in \mathscr{F}$ if $A \subset I$.

A system $\mathscr{F}$ satisfying (i)—(iii) is called a filter.

Definition 2. Let I be a given non-empty set. A non-empty system $\mathscr{B}$ of subsets of the set I is called a basis of a filter on I iff

(i) $\emptyset \notin \mathscr{B}$,

(ii) $A \in \mathscr{B} \wedge B \in \mathscr{B} \Rightarrow A \cap B \in \mathscr{B}$.

By means of Zorn's lemma the following proposition can be proved:

Proposition 1. Let $\mathscr{B}$ be a basis of a filter on I. Then there exists an ultrafilter $\mathscr{F}$ on I such that $\mathscr{B} \subset \mathscr{F}$.

Construction of a hyperset

Let a set A, a non-empty index set I and an ultrafilter $\mathscr{F}$ on I be given.

Let $\mathscr{A}$ be a set of all mappings from I into A; let $\sim$ mean the following equivalence relation:

$$f \sim g \text{ iff } \{i \in I; f(i) = g(i)\} \in \mathscr{F}.$$

Let A^* mean equivalence classes determined by the relation $\sim$, it means $A^* = \mathscr{A}/\sim$. The set A^* is called a hyperset to the set A.

Note. The eqivalence class containing a mapping f will be denoted as $[f]$.

Note. In order to be able to involve A into A^* we identify every element $x \in A$ with that class from A^* which contains the constant mapping $f(i) = x$ for every $i \in I$. Then it is posible to write $A \subset A^*$.

Definition 3. Let A^* be the hyperset to the set A constructed by means of the ultrafilter $\mathscr{F}$ on the index set I; further let $B \subset A$. Then B^*_A is defined as

$$B^*_A = \{\alpha \in A^*; f \in \alpha \Rightarrow \{i \in I; f(i) \in B\} \in \mathscr{F}\}.$$

Note. From properties of an ultrafilter it follows that the definition of B^*_A can be written in the form

$$B^*_A = \{\alpha \in A^*; \exists f \in \alpha: \{i \in I; f(i) \in B\} \in \mathscr{F}\}.$$

From properties of an ultrafilter and Definition 3 we have

Proposition 2. Let A^* be the hyperset to the set A constructed by means of the ultrafilter $\mathscr{F}$ on the index set I; let X and Y be subsets of the set A. Then

(i) $X \subset Y \Rightarrow X^*_A \subset Y^*_A$,

(i) $(X \cup Y)^*_A = X^*_A \cup Y^*_A$,

(iii) $(X \cap Y)^*_A = X^*_A \cap Y^*_A$,

(iv) $(X \setminus Y)^*_A = X^*_A \setminus Y^*_A$,

(v) $(X^c)^*_A = (X^*_A)^c$.

Note. In (v) the complement on the left-hand side is related to A and the complement on the right-hand side is related to A^*.

If an ordered pair $\langle \alpha, \beta \rangle$ for two given elements $\alpha, \beta \in A^*$ is defined as

$$\langle \alpha, \beta \rangle = \{\langle a, b \rangle;\ a \in \alpha \wedge b \in \beta\},$$

then it is not difficult to prove the following

Proposition 3. Let A^* and B^* be hypersets to sets A and B, respectively, constructed by means of the same ultrafilter $\mathscr{F}$ on the same index set I. Then

$$(A \times B)^* = A^* \times B^*.$$

(Let $\alpha \in (A \times B)^*$, $\alpha = [y]$, where y is a mapping $y: I \to A \times B$. Then y can be written as $y = (y_1, y_2)$, where y_1 and y_2 are mappings $y_1: I \to A$ and $y_2: I \to B$ defined in the following way: let $i \in I$, $y(i) = \langle a, b \rangle$; then $y_1(i) = a$, $y_2(i) = b$. Then it holds

$$[y] = \langle [y_1], [y_2] \rangle .$$

($(A \times B)^*$ is assumed to be constructed by means of the ultrafilter $\mathscr{F}$ on the index set I.)

Definition 4. Let X^* and Y^* be hypersets to sets X and Y, respectively, constructed by means of the ultrafilter $\mathscr{F}$ on the index set I. Let S be a binary relation from X into Y. Then the hyperrelation S^* is defined as

$$S^* = \{\langle \alpha, \beta \rangle \in X^* \times Y^*;\ \forall f \in \alpha\ \forall g \in \beta\colon \{i \in I;\ \langle f(i), g(i) \rangle \in S\} \in \mathscr{F}\}.$$

It is not difficult to prove the following two propositions.

Proposition 4. Let f be a mapping from X into Y. Then f^* is a mapping from X^* into Y^* and $f^*/X = f$.

Proposition 5. Let f^* be a hypermapping to the mapping $f\colon X \to Y$. Let $A \subset X$, $B \subset Y$. Then

(i) $f^*(A^*_X) = (f(A))^*_Y$,

(ii) $f^{*^{-1}}(B^*_Y) = (f^{-1}(B))^*_X$.

(All hyperobjects are assumed to be constructed by means of the same ultrafilter $\mathscr{F}$ on the same index set I.)

II. Topological spaces

Definition 5. Let $(X, \mathscr{U})$ be a topological space. Let X^* be the hyperset to the set X constructed by means of the ultrafilter $\mathscr{F}$ on the index set I. Let $x \in X$.

The set

$$\mu(x) = \cap \{G^*_X;\ x \in G \wedge G \in \mathscr{U}\}$$

is called a monad of the element x. If $\alpha \in \mu(x)$, then it can be written $\alpha \approx x$ (α is infinitely near to x).

Note. The binary relation $\approx$ is defined on $X^* \times X$.

It is easy to prove (by means of Proposition 2 (i)) the following

Proposition 6. Let $(X, \mathscr{U})$ be a topological space; let $x \in X$; let $\mathscr{O}_x$ be a basis of the surroundings of x. Then

$$\mu(x) = \cap \{A^*_X;\ A \in \mathscr{O}_x\}.$$

Let $(X, \mathscr{U})$ be a topological space. Now the class $\mathscr{I}_X$ of index sets will be defined. The index set needful to the construction of the hyperset X^* will be selected from the class $\mathscr{I}_X$.

Definition 6. Let X be a given set; let $Z \supset 2^X$. Let I_Z be defined as $I_Z = \{A \subset 2^Z;\ A \text{ is a finite set }\}$. Then the class $\mathscr{I}_X$ is defined as

$$\mathscr{I}_X = \{I_Z;\ Z \supset 2^X\}.$$

On each of index sets I_Z from the class $\mathscr{I}_X$ a basis $\mathscr{B}_Z$ of a filter is defined in the following way:

Definition 7. Let $I_Z \in \mathscr{I}_X$. Then

$$B \in \mathscr{B}_Z \Leftrightarrow B \subset I_Z \wedge \exists n \in N\ \exists G_1 \subset Z \ldots \exists G_n \subset Z:$$
$$B = \{Y \in I_Z;\ G_i \in Y \text{ for } i = 1, \ldots, n\}.$$

With respect to Proposition 1 then there exists an ultrafilter on I containing the basis $\mathscr{B}_Z$. This ultrafilter will be denoted (for a given index set I_Z) as $\mathscr{F}$.

Theorem 1. Let $(X, \mathscr{U})$ be a topological space, let $A \subset X$ and $I \in \mathscr{I}_X$. Then the set A is open iff $\forall x \in Y$: $\mu(x) \subset A^*_X$.

Proof. "$\Rightarrow$" Let A be an open set, let $x \in A$ and let $\mathscr{A}$ be defined as $\mathscr{A} = \{G^*_X;\ x \in G \wedge G \in \mathscr{U}\}$. Then $\mu(x) = \cap \mathscr{A}$, further $A^*_X \subset \mathscr{A}$, therefore $\mu(x) = \cap \mathscr{A} \subset A^*_X$.

"$\Leftarrow$" Indirectly. Let A be a non-open set. Then $\exists x \in A\ \forall G \in \mathscr{U}$: $x \in G \Rightarrow G \setminus A \neq \emptyset$. Now the validity of the statement $\exists x \in A\ \exists \alpha \in X^*$: $\alpha \in \mu(x) \wedge \alpha \notin A^*_X$ will be proved. With respect to the axiom of choice there exists a mapping f: $\{G \in \mathscr{U};\ x \in G\} \to X$ such that $f(G) \in G \setminus A$.

Let now i' and p_i be defined as $i' = i \cap \{G \in \mathscr{U};\ x \in G\}$,
$p_i = \begin{cases} \cap i' \text{ if } i' \neq \emptyset, \\ X \text{ if } i' = \emptyset. \end{cases}$ Let the mapping a: $I \to X$ be defined in the following way: $a(i) = f(p_i)$. Let $\alpha = [a]$.

From the construction of the mapping a it follows: Let $G \in \mathscr{U}$, $x \in G$, then $G \in i \Rightarrow a(i) \in G$, $G \in i \Rightarrow a(i) \notin A$. From these statement it follows: Let $G \in \mathscr{U}$, $x \in G$, then

$$\{i \in I;\ a(i) \in G\} \supset \{i \in I;\ G \in i\},$$
$$\{i \in I;\ a(i) \notin A\} \supset \{i \in I;\ G \in i\}.$$

With respect to Definition 7 it holds: $\{i \in I;\ G \in i\} \in \mathscr{F}$. Thus with respect to (iii) in Definition 1 it holds:

(*) $\forall G \in \mathscr{U}: x \in G \Rightarrow \{i \in I;\ a(i) \in G\} \in \mathscr{F}$,

(*) $\forall G \in \mathscr{U}: x \in G \Rightarrow \{i \in I;\ a(i) \notin A\} \in \mathscr{F}$.

The statement (*) means: $\forall G \in \mathscr{U}: x \in G \Rightarrow \alpha \in G^*_X$, i.e. $\alpha \in \mu(x)$; the statement (**) means $\alpha \notin A^*_X$. Consequently it holds $\exists x \in A\ \exists \alpha \in X^*: \alpha \in \mu(x) \wedge \alpha \notin A^*_X$. Q.E.D.

Thereorem 2. Let $(X, \mathscr{U})$, $(Y, \mathscr{V})$ be topological spaces; let f be a mapping X into Y, let $I \in \mathscr{I}_X \cap \mathscr{I}_Y$. Then f is a continuous mapping iff $\forall x \in X\ \forall \alpha \in X^*$: $\alpha \approx x \Rightarrow f^*(\alpha) \approx f(x)$ ($\approx$ on the left-hand side of the implication means infinitely near in $(X, \mathscr{U})$, $\approx$ on the right-hand side of the implication means infinitely near in $(Y, \mathscr{V})$).

Proof. "$\Rightarrow$" Let f be a continuous mapping X into Y. Then $\forall x \in X$ $\forall V \in \mathscr{V}: f(x) \in V \Rightarrow (x \in f^{-1}(V) \wedge f^{-1}(V) \in \mathscr{U})$. Let $U = f^{-1}(V)$, then $f(U) = V$. With respect to (i) in Proposition 5 $f^*(U^*_X) = V^*_Y$. The set U is open, consequently with respect to Theorem 1 $\forall x \in U$: $\mu(x) \subset U^*_X$, therefore $\forall x \in U$: $f^*(\mu(x)) \subset V^*_Y$. This relation is true for each open set $V \in \mathscr{V}$ such that $f(x) \in V$. Consequently,

$$f^*(\mu(x)) \subset \cap \{V^*_Y;\ V \in \mathscr{V} \wedge f(x) \in V\} = \mu(f(x)).$$

Hence the implication is proved.

"$\Leftarrow$" Let $\forall x \in X$: $f^*(\mu(x)) \subset \mu(f(x))$. The statement $V \in \mathscr{V} \Rightarrow f^{-1}(V) \in \mathscr{U}$ is to be proved. Let $V \in \mathscr{V}$. With respect to (ii) in Proposition 5 $(f^{-1}(V))^*_X = f^{*-1}(V^*_Y)$. With respect to Theorem 1 $\forall y \in V$: $\mu(y) \subset V^*_Y$, since V is an open set in $\mathscr{V}$. Let now x be an arbitrary element of the set $f^{-1}(V)$. With respect to the presumption $\forall x \in X$: $f^*(\mu(x)) \subset \mu(f(x))$ it holds that $x \in f^{-1}(V) \Rightarrow \mu(x) \subset f^{*-1}(V^*_Y) = (f^{-1}(V))^*_X$. With respect to Theorem 1 this statement means that $f^{-1}(V)$ is an open set. Q.E.D.

Theorem 3. Let $(X, \mathscr{U})$ be a topological space, let $A \subset X$ and $I \in \mathscr{I}_X$. Then A is a compact set iff $\forall \alpha \in \mathrm{A}^*_X\ \exists \mathrm{a} \in \mathrm{A}$: $\alpha \approx \mathrm{a}$.

Proof. "$\Rightarrow$" This implication will be proved by a contradiction. Let A a compact set, let there exist $\alpha \in A^*_X$ such that $\forall x \in A$: $\alpha \notin \mu(x)$. As $\mu(x) = \cap \{G^*_X;\ G \in \mathscr{U} \wedge x \in G\}$, from the given assumptions it follows

$$(*) \qquad \forall x \in A\ \exists G(x) \in \mathscr{U}: x \in G(x) \wedge \alpha \notin (G(x))^*_X.$$

The system $\{G(x);\ x \in A\}$ is an open covering of the compact set A, therefore there exist elements $x_1, \ldots, x_n$ such that $A \subset G(x_1) \cup \ldots \cup G(x_n)$. With respect to Proposition 2 (i) and (ii) (extended by the mathematical induction) $A^*_X \subset G(x_1)^*_X \cup \ldots \cup G(x_n)^*_X$. As $\alpha \in A^*_X$, there have to exist $j \in \{1, \ldots, n\}$ such that $\alpha \in (G(x_j))^*_X$, which is contradictory to (*).

"$\Leftarrow$" This implication will be proved by a contradiction, too. Let A be a non-compact set and

$$(**) \qquad \forall \alpha \in A_X^* \, \exists x \in A: \alpha \approx x .$$

Then there exists such an open convering $\mathscr{C}$ of the set A that each of its finite subsystems is not an open covering of A. By means of the axiom of choice it is possible to construct a mapping f: $\{\mathscr{A} \subset \mathscr{C}; \mathscr{A}$ is a finite non-empty subsystem of $\mathscr{C}\} \to X$ such that $f(\mathscr{A}) \in A \setminus \cup \mathscr{A}$.

Now an element $\beta \in A_X^*$ such that $\forall C \in \mathscr{C}: \beta \notin C_X^*$ will be constructed.

Let for each $i \in I$, i' be defined in the following way: $i' = i \cap \mathscr{C}$. Let $y: I \to A$ be a mapping

$$y(i) = \begin{cases} f(i') & \text{if } i' \neq \emptyset, \\ a & \text{if } i' = \emptyset, \end{cases}$$

here a is a fixed element of A. Let $\beta = [y]$. Evidently, $\beta \in A_X^*$. With respect to the construction of the mapping y

$$\forall i \in I \, \forall C \in \mathscr{C}: C \in i \to y(i) \notin C ,$$

therefore $\forall C \in \mathscr{C}$: $\{i \in I; \ y(i) \notin C\} \supset \{i \in I; \ C \in i\}$. The set $\{i \in I; \ C \in i\}$ is an element of the basis $\mathscr{B}$ of the filter on I, further $\mathscr{B} \subset \mathscr{F}$, therefore $\{i \in I; C \in i\} \in \mathscr{F}$. With respect to the property (iii) of an ultrafilter $\{i \in I; y(i) \notin C\} \in \mathscr{F}$, which means

$$(***) \qquad \forall C \in \mathscr{C}: \beta \notin C_X^* .$$

With respect to $(**)$ there exists $x_\beta \in A$ such that $\beta \approx x_\beta$. The system $\mathscr{C}$ is an open covering of A, therefore there exists $C(x_\beta) \in \mathscr{C}$ such that $x_\beta \in C(x_\beta)$. The set $C(x_\beta)$ is open, therefore $\mu(x_\beta) \subset (C(x_\beta))_X^*$. It holds $\beta \in \mu(x_\beta)$, therefore $\beta \in (C(x_\beta))_X^*$, which is contradictory to $(***)$. Thus the implication is proved.

Theorem 4. Let $(X, \mathscr{U})$ be a product topological space of spaces (X_j, U_j) $(j \in J)$, let $I \in \mathscr{I}_X \cap \bigcap_{j \in J} \mathscr{I}_{X_j}$. Let π_j $(j \in J)$ denote the projections $\pi_j: X \to X_j$, let x_j and α_j be defined as $x_j = \pi_j(x)$ and $\alpha_j = \pi^*(\alpha)$ for each $x \in X$ and each $\alpha \in X^*$. Then it holds $\forall x \in X \, \forall \alpha \in X^*$: $\alpha \in \mu(x) \Leftrightarrow \forall j \in J$: $\alpha_j \in \mu(x_j)$.

Proof: "$\Rightarrow$" In a product topology each of the projections is a continuous mapping. Then with respect to Theorem 2

$$\forall x \in X \, \forall \alpha \in X^*: \alpha \in \mu(x) \Rightarrow \forall j \in J: \alpha_j \in \mu(x_j) .$$

"$\Leftarrow$" Let us begin with a consideration: Let the subbasis $\mathscr{S}$ of the topology $\mathscr{U}$ be defined as $\mathscr{S} = \{\pi_j^{-1}(A); j \in J \wedge A \in \mathscr{U}_j\}$. Let the basis $\mathscr{D}$ of the topology $\mathscr{U}$ be defined as the system of all finite intersections of sets from $\mathscr{S}$. Let us investigate the relation $\alpha \in G_X^*$, where $G \in \mathscr{D}$. If $G \in \mathscr{D}$, then there exist sets $G_{j_1} \in \mathscr{U}_{j_1}, \ldots, G_{j_n} \in \mathscr{U}_{j_n}$ such that

$$(*) \qquad G = \{z \in X;\ z_{j_k} \in G_{j_k} \quad \text{for} \quad k = 1, \ldots, n\}.$$

Let $y \in \alpha$ and $\alpha \in X^*$. With respect to the note following Definition 3 it holds $\alpha \in G^*_X$ iff $\{i \in I;\ y(i) \in G\} \in \mathscr{F}$ ($\mathscr{F}$ is the ultrafilter from Definition 7). With respect to $(*)$ it means

$$(**) \qquad \alpha \in G^*_X \Leftrightarrow \{i \in I;\ (y(i))_{j_k} \in G_{j_k} \quad \text{for} \quad k = 1, \ldots, n\} \in \mathscr{F}.$$

Now we give the proof of our implication. Let the following statements hold: $x \in X$; $\alpha \in X^*$; $\forall j \in J$: $\alpha_j \in \mu(x_j)$. We have to prove: $\alpha \in \mu(x)$. As the system $\{G \in \mathscr{D};\ x \in G\}$ is a basis of surroundings of x, with respect to Proposition 6 it is sufficient to prove the statement

$$\forall G \in \mathscr{D}:\ x \in G \Rightarrow \alpha \in G^*_X.$$

Let $G \in \mathscr{D}$, $x \in G$. Then there exists sets $G_{j_1} \in \mathscr{U}_{j_1}, \ldots, G_{j_n} \in \mathscr{U}_{j_n}$ such that $G = \{z \in X;\ z_{j_k} \in G_{j_k}$ for each $k \in \{1, \ldots, n\}\}$. As $x \in G$, it holds $\forall k \in \{1, \ldots, n\}$: $x_{j_k} \in G_{j_k}$. As the sets $G_{j_1}, \ldots, G_{j_n}$ are open in the proper topologies, it holds $\forall k \in \{1, \ldots, n\}$: $\mu(x_{j_k}) \subset (G_{j_k})^*_X$. From the assumption: $\alpha_j \in \mu(x_j)$ for each $j \in J$, it follows

$$\forall k \in \{1, \ldots, n\}:\ \{i \in I;\ (y(i))_{j_k} \in G_{j_k}\} \in \mathscr{F}.$$

As a finite intersection of sets from $\mathscr{F}$ is a set from $\mathscr{F}$, it holds $\bigcap_{k=1}^{n} \{i \in I;\ (y(i))_{j_k} \in G_{j_k}\} \in \mathscr{F}$. Consequently, $\{i \in I;\ (y(i))_{j_k} \in G_{j_k}$ for $k = 1, \ldots, n\} \in \mathscr{F}$, which with respect to $(**)$ means: $\alpha \in G^*_X$, q.e.d.

III. Uniform spaces

Proposition 7. Let $(X, \mathscr{R})$ be a uniform space, let $\mathscr{U}_{\mathscr{R}}$ be the proper uniform topology, let I be an arbitrary non-empty index set, let $\mathscr{G}$ be an ultrafilter on I.

Let the binary relation $S \subset X^* \times X^*$ be defined in the following way

$$\langle \alpha, \beta \rangle \in S \Leftrightarrow \forall f \in \alpha\ \forall g \in \beta\ \forall U \in \mathscr{R}:\ \{i \in I;\ \langle f(i), g(i) \rangle \in U\} \in \mathscr{G}.$$

Then the relation S is an extension of the relation $\approx$ of the infinite nearness ($\approx\ = \{\langle \alpha, x \rangle \in X^* \times X;\ \alpha \approx x\}$), i.e. it holds

$$\forall \alpha \in X^*\ \forall x \in X:\ \langle \alpha, x \rangle \in S \Leftrightarrow \alpha \approx x.$$

(**Note.** It is not difficult to prove that it holds (similarly as by the relation $\approx$):

$$\langle \alpha, \beta \rangle \in S \Leftrightarrow \exists f \in \alpha\ \exists g \in \beta:\ \forall U \in \mathscr{R}:\ \{i \in I;\ \langle f(i), g(i) \rangle \in U\} \in \mathscr{G}.$$

This statement is true, since α and β are defined as equivalence classes.)

Proof: "$\Rightarrow$" Let $\langle \alpha, x\rangle \in S, f \in \alpha$; that means

(*) $$\forall U \in \mathcal{R}: \{i \in I;\ \langle f(i), x\rangle \in U\} \in \mathcal{G}.$$

It is necessary to prove the following implication

$$\forall A \in U_{\mathcal{R}}: x \in A \Rightarrow \alpha \in A^*_X.$$

With respect to the construction of the uniform topology $\mathcal{U}_{\mathcal{R}}$

$$A \in \mathcal{U}_{\mathcal{R}} \wedge x \in A \Rightarrow \exists U \in \mathcal{R}: U(x) \subset A,$$

where $U(x) = \{y \in X: \langle x, y\rangle \in U\}$. Then $\{i \in I; f(i) \in A\} \supset \{i \in I; f(i) \in U(x)\} = \{i \in I; \langle x, f(i)\rangle \in U\} = \{i \in I; \langle f(i), x\rangle \in U^{-1}\}$. As $U^{-1} \in \mathcal{R}$ (with respect to the definition of a uniform space), with respect to (*)

$$\{i \in I; f(i) \in A\} \supset \{i \in I;\ \langle f(i), x\rangle \in U^{-1}\} \in \mathcal{G},$$

therefore $\{i \in I; f(i) \in A\} \in \mathcal{G}$, which means $\alpha \in A^*_X$, q.e.d.

"$\Leftarrow$" Let $\alpha \approx x$ and $f \in \alpha$. The system $\{U(x);\ U \in \mathcal{R}\}$ (where $U(x) = \{y \in X; \langle x, y\rangle \in U\}$) is in the uniform topology $\mathcal{U}_{\mathcal{R}}$ a basis of surroundings of the element x. With respect to Proposition 6, consequently $\mu(x) = \cap \{(U(x))^*_X$ $U \in \mathcal{R}\}$. As $\alpha \in \mu(x)$, it holds

(**) $$\forall U \in \mathcal{R}: \{i \in I; f(i) \in U(x)\} \in \mathcal{G}.$$

It is necessary to prove the following statement

$$\forall V \in \mathcal{R}: \{i \in I;\ \langle f(i), x\rangle \in V\} \in \mathcal{G}.$$

Its validity follows from the equalities

$$\{i \in I;\ \langle f(i), x\rangle \in V\} = \{i \in I;\ \langle x, f(i)\rangle \in V^{-1}\} = \{i \in I; f(i) \in V^{-1}(x)\},$$

from the statement (**) and from the fact that $V^{-1} \in \mathcal{R}$.

Proposition 7 gives a reson for using the symbol $\approx$ in the following definition:

Definition 8. Let $(X, \mathcal{R})$ be a uniform space, let I be an arbitrary non-empty index set, let $\mathcal{G}$ be an ultrafilter on I. Let $\alpha, \beta \in X^*$. We say that the element α is infinitely near to the element $\beta (\alpha \approx \beta)$ iff

$$\forall f \in \alpha\ \forall g \in \beta\ \forall U \in \mathcal{R}: \{i \in I;\ \langle f(i), g(i)\rangle \in U \in \mathcal{G}.$$

It is not difficult to prove the following

Proposition 8. Let $(X, \mathcal{R})$ be a uniform space, let I be an arbitrary non-empty index set, let $\mathcal{G}$ be an ultrafilter on I. Then the relation $\{\langle \alpha, \beta\rangle \in X^* \times X^*;\ \alpha \approx \beta\}$ is reflexive, symmetrical and transitive.

Theorem 5. Let $(X, \mathscr{R})$ and $(Y, \mathscr{S})$ be uniform spaces, let f be a mapping X into Y, let $I \in \mathscr{I}_X \cap \mathscr{I}_{X \times X} \cap \mathscr{I}_Y$. Then the mapping f is uniformly continuous iff

$$\forall \alpha \in X^* \ \forall \beta \in X^*: \ \alpha \approx \beta \Rightarrow f^*(\alpha) \approx f^*(\beta).$$

Proof: " $\Rightarrow$ " Let f be a uniformly continuous mapping, let $\alpha \in X^*n$, $\beta \in X^*$, $a \in \alpha$, $b \in \beta$, $\alpha \approx \beta$. Consequently,

$$(*) \qquad \forall U \in \mathscr{R}: \{i \in I; \ \langle a(i), b(i) \rangle \in U\} \in \mathscr{F}.$$

The composite mappings $f(a(.))$ and $f(b(.))$ are elements of the classes $f^*(\alpha)$ and $f^*(\beta)$, respectively. With respect to the note following Proposition 7 it is sufficient to prove the statement

$$\forall V \in \mathscr{S}: \{i \in I; \ \langle f(a(i), f(b(i)) \rangle \in V\} \in \mathscr{F}.$$

Let $V \in \mathscr{S}$; with respect to the definition of a uniformly continuous mapping

$$\{\langle x, y \rangle \in X \times X; \ \langle f(x), f(y) \rangle \in V\} \in \mathscr{R}.$$

With respect to $(*)$ then it holds $\{i \in I; \ \langle f(a(i)), f(b(i)) \rangle \in V\} \in \mathscr{F}$, q.e.d.

" $\Leftarrow$ " This implication will be proved by a contradiction. Let f not be a uniformly continous mapping; then it holds

$$(**) \qquad \exists V \in \mathscr{S}: \forall U \in \mathscr{R} \ \exists \langle x_U, y_U \rangle \in U: \ \langle f(x_U), f(y_U) \rangle \notin V.$$

Now an ordered pair $\langle \alpha, \beta \rangle \in X^* \times X^*$ such that $\alpha \approx \beta$ and $f^*(\alpha) \not\approx f^*(\beta)$ will be constructed. With respect to the axiom of choice and $(**)$ there exists a mapping $r: \mathscr{R} \to X \times X$ such that

$$\forall U \in \mathscr{R}: \ r(U) \in U \wedge f(r(U)) \notin V$$

($f(r(U))$ means that the mapping will be applied to both parts of $r(U)$).

If i' and p_i are defined as $i' = i \cap \mathscr{R}$ and $p_i = \begin{cases} \cap i' & \text{if } i' \neq \emptyset, \\ X \times X & \text{if } i' = \emptyset, \end{cases}$ then a mapping $y: I \to X \times X$ can be defined in the following way

$$y(i) = r(p_i), \quad i \in I.$$

It holds $[y] \in (X \times X)^*$, with respect to Proposition 3 $[y] = \langle [y_1], [y_2] \rangle$, where the mappings $y_1: I \to X$ and $y_2: I \to X$ are defined in the following way: Let $i \in I$, $y(i) = \langle a, b \rangle$, then $y_1(i) = a$, $y_2(i) = a$. Let α and β be defined as $\alpha = [y_1]$, $\beta = [y_2]$. With respect to the construction of the mapping y it holds: Let $i \in I$, $U \in \mathscr{R}$, then $U \in i \Rightarrow \langle y_1(i), \ y_2(i) \rangle \in U$. It means $\{i \in I; \ \langle y_1(i), \ y_2(i) \rangle \in U\} \supset \supset \{i \in I; \ U \in i\}$. The set $\{i \in I; \ U \in i\}$ is an element of the basis $\mathscr{B}$ of the filter from Definition 7 and as $\mathscr{B} \subset \mathscr{F}$, it holds $\{i \in I; \ U \in i\} \in \mathscr{F}$ and consequently $\{i \in I; \langle y_1(i), \ y_2(i) \rangle \in U\} \in \mathscr{F}$. Then $\forall U \in \mathscr{R}: \{i \in I; \ \langle y_1(i), \ y_2(i) \rangle \in U\} \in \mathscr{F}$, which with respect to the note in Proposition 7 is sufficient for $\alpha \approx \beta$. Further from the construction of the mapping y it follows $\forall i \in I$: $\langle f(y_1(i)), f(y_2(i))\rangle \notin V$. Consequently,

$$(***) \qquad \{i \in I;\ \langle f(y_1(i)), f(y_2(i))\rangle \in V\} = \emptyset \notin \mathscr{F}\,.$$

As $f^*(\alpha) = f(y_1(.))$, $f^*(\beta) = f(y_2(.))$, (***) stands for $f^*(\alpha) \notin f^*(\beta)$, q.e.d.

IV. Two simple applications

1. The product topological space of compact spaces is compact.

Proof. Let $(X, \mathscr{U})$ be a product topological space of compact spaces $(X_j, \mathscr{U}_j)$ $(j \in J)$. Theorem 3 and Theorem 4 will be used. Let $\alpha \in X^*$, α_j let be defined as $\alpha_j = \pi_j^*(\alpha)$, consequently $\alpha_j \in X_j^*$. As X_j is a compact space, there exists $x_j \in X_j$ such that $\alpha_j \approx x_j$. Using the axiom of choice a mapping x: $J \to \bigcup_{i \in J} X_j$ will be constructed such that $\alpha_j \approx x(j)$ for each $j \in J$. With respect to Theorem 4 $\alpha \approx x$. Consequently it holds $\forall \alpha \in X^*\ \exists x \in X$: $\alpha \approx x$, which means X is a compact space.

2. Let $(X, \mathscr{R})$ and $(Y, \mathscr{S})$ be uniform spaces, let the topological space $(X, \mathscr{U}_{\mathscr{R}})$ be compact, let f be a continuous mapping f: $X \to Y$. Then f is a uniformly continuous mapping.

Proof. Theorems 2, 3, 5 and Proposition 8 will be used. There exists $a \in X$ such that $\alpha \approx a$, since X is a compact space. The relation $\approx$ is symmetrical and transitive, thus $\beta \approx \alpha \wedge \alpha \approx a \Rightarrow \beta \approx a$. As f is a continuous mapping

$$\alpha \approx a \Rightarrow f^*(\alpha) \approx f(a)$$

$$\beta \approx a \Rightarrow f^*(\beta) \approx f(a)\,.$$

Finally, $f^*(\alpha) \approx f(a) \wedge f(a) \approx f^*(\beta) \Rightarrow f^*(\alpha) \approx f^*(\beta)$,

which means f is a uniformly continuous mapping.

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РЕЗЮМЕ

ЭЛЕМЕНТАРНЫЙ НЕСТАНДАРТНЫЙ ПОДХОД К ТОПОЛОГИЧЕСКИМ ПРОСТРАНСТВАМ

Збынек Кубачек, Братислава

В статье показывается возможность выбора конкретного множества индексов для конструкции нестандартных элементов в топологических пространствах и даётся доказательство нестандартного описания некоторых топологических понятий.

SÚHRN

ELEMENTÁRNY NEŠTANDARDNÝ PRÍSTUP K TOPOLOGICKÝM PRIESTOROM

Zbyněk Kubáček, Bratislava

Práca ukazuje možnosť výberu konkrétnej indexovej množiny potrebnej pre konštrukciu neštandardných prvkov v topologických priestoroch a dôkaz neštandardného popisu niektorých topologických pojmov.