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VECTOR-COVERING SYSTEMS WITH FOUR EQUAL MODULI

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1. Preliminary facts

Let a and n be integers with $0 \le a < n$. By $a(n)$ denote the set of the all numbers of the form $a + sn$, where s is an integer. Let f be the characteristic function of the set $a(n)$ on Z.

Definition. Let a vector $\varepsilon = (v_1, \ldots, v_m)$ with rational v_j' s be given. The system

$$(1) \qquad a_j(n_j), \quad j = 1, \ldots, m, \quad 2 \le n_1 \le \ldots \le n_m$$

will be called an ε-covering or vector-covering, abbreviated VCS, if for any $r \in Z$ we have

$$\sum_{j=1}^{m} v_j f_j(r) = 1$$

This kind of covering systems was introduced in 1974 by Š Znám [7].

Definition. A non-empty system (1) is called ε-vanishing if for all $r \in Z$ we have

$$\sum_{j=1}^{n} v_j f_j(r) = 0$$

A VCS (1) with distinct congruences and with $v_j \neq 0$ for every $j = 1, \ldots, m$ is called reduced if it does not contain any non-empty vanishing subsystem.

The following three lemmas give some basic properties of vanishing systems.

Lemma 1 [8]. If (1) is an ε-vanishing system, then for all $j = 1, \ldots, m$ and $s = 1, \ldots, n_j$ we have

$$\sum_{\substack{t=1 \\ n_j \backslash sn_t}}^{m} \frac{v_t}{n_t} \exp \frac{2\pi i s a_t}{n_j} = 0$$

Lemma 2 [8]. If (1) is an ε-vanishing and $v_j \neq 0$ for every j, then we have $n_{m-1} = n_m$.

Lemma 3 [8]. Let (1) be ε-vanishing with all $v_j \neq 0$ and $n_1 < n_2 \ldots < n_{m-2} < < n_{m-1} = n_m$, then for all $j = 1, \ldots, m$ we have $n_j \backslash n_m$.

The special case of VCS for $\varepsilon = (1, \ldots, 1)$ is called disjoint covering system, abbreviated DCS. DCS's were studied first by Paul Erdös [1].

Davenport, Mirski, Newman and Radó showed the following interesting theorem. If (1) is a DCS, then at least two of the moduli are equal, namely $n_{m-1} = n_m$.

In 1957 Stein [5] studied the case if in a DCS there exists exactly one couple of equal moduli. He proved:

Let (1) be a DCS. If $n_1 < n_2 < \ldots < n_{m-2} < n_{m-1} = n_m$ then we have $n_j = 2^j$ for $j = 1, \ldots, m-2$ and $n_{m-1} = n_m = 2^{m-1}$.

In 1969 Š. Znám solved the case in which there exists exactly one triple of equal moduli and the remaining ones being distinct. He proved [6]:

If (1) is a DCS with $n_1 < n_2 < \ldots < n_{m-3} < n_{m-2} = n_{m-1} = n_m$ then the moduli are uniquely determined and $n_j = 2^j$ for $j = 1, \ldots, m-3$, while $n_{m-2} = = n_{m-1} = n_m = 3 \cdot 2^{m-3}$.

In 1972 Š. Porubský [4] proved the following theorem:

Let (1) be a DCS. If $n_1 < n_2 < \ldots < n_{m-4} < n_{m-3} = n_{m-2} = n_{m-1} = n_m$ then there are two possibilities:

a) $n_j = 2^j$ for $j = 1, \ldots, m-4$, $\quad n_{m-3} = n_{m-2} = n_{m-1} = n_m = 2^{m-2}$,

b) $n_j = 2^j$ for $j = 1, \ldots, m-5$, $\quad n_{m-4} = 3 \cdot 2^{m-5}$, $n_{m-3} = \ldots = n_m = 3 \cdot 2^{m-4}$.

Theseresults are generalized for the case of VCS as follows:

Theorem 1 [7]. If (1) is a $(v_1, \ldots, v_m)$-covering system with all $v_j \neq 0$ in which there exist exactly two equal moduli then (1) is a DCS and consequently $n_j = 2^j$ for $j = 1, \ldots, m-2$, $n_{m-1} = n_m = 2^{m-1}$ holds.

Theorem 2 [8]. Let (1) be a reduced VCS with either one triple or two couples of equal moduli whereas the others are distinct. Then all its moduli are of the form $n_j = 3^a \cdot 2^b$, where $a = 0$ or 1 and b is nonnegative integer.

In our considerations we shall further use the following result of Mann [2].

Consider the relation

$$\sum_{t=1}^{r} a_t \zeta_t = 0 \tag{2}$$

where a_t are rational integers and ζ_t are roots of unity. If the left-hand side of (2) is irreducible i.e. if none of its proper subsums is equal to 0, then there exist such primes $p_1 < \ldots < p_s \leq r$ and such $p_1 p_2 \ldots p_s$-th roots of unity $\eta_t (t = 1, \ldots, r)$ that

$$\zeta_t = \alpha \cdot \eta_t$$

where α is a fixed number. If moreover, all a_t's are positive and we cannot choose $p_s < r$, then we have $s = 1$ and the numbers a_t coincide.

Theorem 3 [8]. If (1) is an ε-covering system, the for all $j = 1, \dots, m$ we have

$$\sum_{\substack{t=1 \\ n_j \backslash sn_t}}^{m} \frac{v_t}{n_t} \exp \frac{2\pi s a_t}{n_j} = \begin{cases} 0 & \text{if} \quad s = 1, \dots, h_j - 1 \\ 1 & \text{if} \quad s = n_j . \end{cases}$$

2.Lemma

In what follows we shall always suppose that a, b, c, d are noonnegative integers. To prove the main result we shall need the following result.

Lemma 4. Let (1) be a ε-vanishing system with all congruences distinct and all $v_j \neq 0$.

a) If (1) contains two couples of congrences with respect to equal moduli, say $a_{t1}(n_0)$, $a_{t2}(n_0)$ and $a_{m-1}(n_m)$, $a_m(n_m)$, then for all $j = 1, \dots, m$

$$n_j \backslash 2^b \cdot n_m \quad \text{and} \quad n_j \backslash 2^c \cdot n_0$$

holds for some nonnegative integers b, c.

b) If (1) contains exactly one triple of congruences with respect to equal moduli, then for all $j = 1, \dots, m$

$$n_j \backslash 2^b \cdot n_m$$

holds for some nonnegative integer b.

Proof.

Recal that according to Lemma 2 the greatest modulus appears at least twice and therefore there is no loss of generality if we suppose that one of the couples is with respect to the greatest modulus. We shall proceed by induction concerning the number m of congruences. Obviously we have $m \geq 3$.

2.1.

If $m = 3$ then we have three distinct congrences $a_1(n)$, $a_2(n)$, $a_3(n)$ with $0 \leq a_1, a_2, a_3 < n$. From the definition of the vanishing system for all $r \in Z$

$$\sum_{j=1}^{3} v_j f_j(r) = 0 \tag{3}$$

holds. The three congruences are distinct, hence they are disjoint. Thus from (3) $v_1 = v_2 = v_3 = 0$ follows. This is a contradiction with the hypotheses of our lemma. Therefore such a vanishing system cannot exist.

2.2.

Now suppose that $m > 3$ and that the assertion of Lemma 4 is true for all systems with less than m congrences. We shall distinquish two cases.

2.2.1.

Let here exist two couples of equal moduli $n_0 \in \{n_1, \ldots, n_{m-2}\}$, $n_{m-1} = n_m$ in the vanishing system. Lemma 1 gives for $j = m$

$$v_{m-1} \exp \frac{2\pi i a_{m-1}}{n_m} + v_m \exp \frac{2\pi i a_m}{n_m} = 0 \tag{4}$$

and consequently $|v_{m-1}| = |v_m|$.

If $v_{m-1} = -v_m$, then (4) yields $\frac{2\pi i a_m}{n_m} = \frac{2\pi i a_{m-1}}{n_m} + 2\pi i c$ that is $a_{m-1} \equiv a_m$ (mod n_m), which contradict that all the congruences are distinct.

If $v_{m-1} = n_m$, then (4) yields $\frac{2\pi i a_m}{n_m} = \frac{2\pi i a_{m-1}}{n_m} + 2\pi i \left(c + \frac{1}{2}\right)$ and $a_m \equiv$ $\equiv a_{m-1} + \frac{n_m}{2}$ (mod n_m). Hence the conguences $a_m(n_m)$, $a_{m-1}(n_m)$ can be replaced by a single congruence $a_{m-1}\left(\frac{n_m}{2}\right)$. In such a way we obtain a vanishing system having $m - 1$ congruences. Consider the following three cases.

a) There is a $n_k \in \{n_1, \ldots, n_{m-2}\} - \{n_0\}$ such that $\frac{n_m}{2} = n_k$. Then we obtain a system with exactly two couples of equal moduli. Thus using the induction assumption, for all $j \neq m$ we have $n_j \backslash 2^b \cdot n_0$ and $\left.\frac{n_m}{2}\right| 2^c \cdot n_0$ and for all j we have $n_j \backslash 2^d \cdot \frac{n_m}{2}$.

b) If for all $j = 1, \ldots, m - 2$ the inequality $\frac{n_m}{2} \neq n_j$ holds, then we have only one couple of equal moduli. Owing to Lemma 3 for all $j = 1, \ldots, m - 2$, $n_j \backslash n_0$ and $\left.\frac{n_m}{2}\right| n_0$ holds. Due to Lemma 1 we obtain

$$v_{t1} \cdot \exp \frac{2\pi i a_{t1}}{n_0} + v_{t2} \cdot \exp \frac{2\pi i a_{t2}}{n_0} = 0$$

and $|v_{t1}| = |v_{t2}|$. If $v_{t1} = -v_{t2}$, then $a_{t1} \equiv a_{t2}$ (mod n_0) which is impossible because all the congruences are distinct. Therefore $v_{t1} = v_{t2}$ and the congruences $a_{t1}(n_0)$, $a_{t2}(n_0)$ can be replaced by a single congruence $a_{t1}\left(\frac{n_0}{2}\right)$. Lemma 2 implies that there exisst $n_j \in \left\{n_1, \dots, n_{m-2}, \frac{n_m}{2}\right\} - \{n_0\}$ so that $n_j = \frac{n_0}{2}$ and that $\frac{n_m}{2} \leq \frac{n_0}{2}$, which yields $n_m \leq n_0$. This is a contradiction with the assumption of our lemma.

c) If $n_0 = \frac{n_m}{2}$, then we have one triple of equal moduli. Owing to the induction assumption for all $n_j \in \{n_1, \dots, n_{m-2}\} - \{n_0\}$ we have

$$n_j \backslash 2^b \cdot n_0 \quad \text{and} \quad n_j \backslash 2^b \cdot \frac{n_m}{2}.$$

2.2.2.

Let one triple of equal moduli in the vanishing system.exist. The last three congruences with equal moduli are $a_{m-2}(n_m)$, $a_{m-1}(n_m)$, $a_m(n_m)$. Lema 1 yields

$$\sum_{j=m-2}^{m} v_j \cdot \exp\frac{2\pi i a_j}{n_m} = 0. \tag{5}$$

Because the v_j's are rational, (5) is of the form (2). The only primes not exceeding $r = 3$ are 2 and 3, thus according to Mann's theorem there exist such 6-th roots of unity η_t that

$$\zeta_t = \exp\frac{2\pi i a_t}{n_m} = \alpha \cdot \eta_t, \quad t = m, m-1, m-2$$

where α is fixed. Therefore the equality (5) can be rewritten as follows:

$$v_{m-2}\eta_{m-2} + v_{m-1}\eta_{m-1} + v_m\eta_m = 0$$

where the numbers η_t are from the set $\left\{\exp\frac{k\pi i}{3}; k = 1, \dots, 6\right\}$. Considering these numbers η_t as vectors in the plane and using elementary geometrical considerations we can easily show that their linear combination with real coefficients can be 0 only in the following cases:

I. All η_t's are parallel. Then at least two of them are equal which is impossible, because we have supposed that the congruences are distinct.

II. The vectors η_t's are edges of a directed equalilateral triangle. Then after a suitable permutation of numbersa_{m-2}, a_{m-1}, a_m we get

$$\frac{2\pi i a_{m-1}}{n_m} = \frac{2\pi i a_m}{n_m} + \frac{2\pi i}{3} \quad \text{thus} \quad a_{m-1} = a_m + \frac{n_m}{3}$$

$$\frac{2\pi i a_{m-2}}{n_m} = \frac{2\pi i a_m}{n_m} + \frac{4\pi i}{3} \quad \text{thus} \quad a_{m-2} = a_m + \frac{2n_m}{3}.$$

This implies that the last three congruences in our vanishing system are

$$a_m + \frac{n_m}{3}(n_m), \quad a_m + \frac{2n_m}{3}(n_m), \quad a_m(n_m) \tag{6}$$

and (5) implies $v_{m-2} = v_{m-1} = v_m$. Replacing the congruences (6) in the vanishing system by a single congruence $a_m\left(\frac{n_m}{3}\right)$ we obtain a $(v_1, \ldots, v_{m-3}, v_m)$-vanishing system. Lemma 2 implies that there exist $j \in \{1, \ldots, m-3\}$ so that $n_j = \frac{n_m}{3}$ and according to Lemma 3 for all $j = 1, \ldots, m-3$ we have $n_j \,\Big|\, \frac{n_m}{3}$. The assertion of Lemma 4 follows in this case.

III. The vectors $\eta_i's$ are edges of an equilateral triangle in which one edge is directed oppositely to the others. We have $|v_{m-2}| = |v_{m-1}| = |v_m|$ however one of these numbers — say v_m — has the sign opposite to the others. Add to the vanishing system two new equal congruences

$$a_{m+1}(n_{m+1}) = a_{m+2}(n_{m+2}) = a_m + \frac{n_m}{2}(n_m)$$

and put $v_{m+1} = v_m$, $v_{m+2} = -v_m$. Obviously

$$a_j(n_j), \quad j = 1, \ldots, m+2$$

is a $(v_1, \ldots, v_{m+2})$-vanishing system. Due to $\exp\frac{2\pi i\left(a_m + \frac{n_m}{2}\right)}{n_m} = -\exp\frac{2\pi i a_m}{n_m}$ (5) yields the equation

$$v_{m-2}\exp\frac{2\pi i a_{m-2}}{n_m} + v_{m-1}\exp\frac{2\pi i a_{m-1}}{n_m} + v_{m+2}\exp\frac{2\pi i a_{m+2}}{n_m} = 0.$$

The vectors appearing here are edges of a directed equilateral triangle and $v_{m-1} = v_{m-2} = v_{m+2}$, thus by the same considerations as in case II. we get that the congruences $a_{m-2}(n_m)$, $a_{m-1}(n_m)$, $a_{m+2}\,(n_m)$ can be replaced by a single congruence $a_{m-2}\left(\frac{n_m}{3}\right)$. By a similar argument it can be shown that the congruences $a_m(n_m)$, $(a_{m+1}(n_{m+1})$ can be replaced by a single congruence $a_m\left(\frac{n_m}{2}\right)$. In this way we obtain a new $(v_1, \ldots, v_{m-2}, v_m)$-vanishing system

$$(7) \qquad a_1(n_1), \ldots, a_{m-3}(n_{m-3}), a_{m-2}\left(\frac{n_m}{3}\right), a_m\left(\frac{n_m}{2}\right).$$

The system (7) consist of $m-1$ congruences. Due to Lemma 2 $n_{m-3} = \frac{n_m}{2}$. Let us distinquish 2 cases.

a) There exist $j \in \{1, \ldots, m-4\}$ such that $n_j = \frac{n_m}{3}$. Owing to the induction hypothesis for all $j = 1, \ldots, m-3$ we have

$$n_j \backslash 2^b \cdot \frac{n_m}{2} \quad \text{and} \quad n_j \backslash 2^c \cdot \frac{n_m}{3}.$$

b) For all $k = 1, \ldots, m-3$ the inequality $n_k \neq \frac{n_m}{3}$ holds. Lemma 3 imples $n_k \left| \frac{n_m}{2} \right.$ for all $k = 1, \ldots, m-3$ and $\frac{n_m}{3} \left| \frac{n_m}{2} \right.$ which is impossible.

The proof of Lema 4 is complete.

3. The main theorem and remarks

Now we shall generalize the result of Porubský.

Theorem 4. Let (1) be a reduced VCS. Let one of the following possibilities appear:

a) (1) contains exactly four congruences with respect to the same modulus

b) (1) contains one triple and one couple of equal moduli

c) (1) contains three couples of equal moduli

whereas the others modulu are distinct in all three case.

Then all moduli are of the form $n_j = 3^a \cdot 2^b$, where $a \in \{0, 1, 2\}$ and b is nonnegative integer.

Remark 1. Obviously we have $m \geq 4$. We shall proceed by induction concerning the number m of congruences. The proof is put in parts 4 and 5. The system is reduced, thus all $v_j \neq 0$. We say that a vanishing subsystem S of VCS is maximal (abbreviated MVS) if deleting S from VCS we get a reduced one.

Remark 2. In every reduced VCS or vanishing system the greatest two moduli are equal. Therefore a vanishing subsystem can exist only in such a system which contains at least two couples of equal moduli, one couple will belong to the vanishing system and the second one will belong to the remaining — reduced — system. Thus if there exist 2 or 3 equal moduli in a system, then there cannot exist a vanishing subsystem and the system is reduced.

4. The first step of induction

Put $m = 4$. Theorem 3 implies that

$$\sum_{j=1}^{4} v_j \exp \frac{2\pi i a_j}{n_m} = 0 . \tag{8}$$

According to Mann's theorem there exist such 6-th roots of unity η_t that

$$\zeta_t := \exp \frac{2\pi i a_t}{n_m} = \alpha \cdot \eta_t , \quad t = 1, 2, 3, 4$$

with α fixed (as in part 2.2.2.). Thus

$$\eta_t \in \left\{ \exp \frac{k\pi i}{3} ; \quad k = 1, \dots, 6 \right\} . \tag{9}$$

All η_t's are distinct, because we supposed that $0 \leq a_1, a_2, a_3, a_4 < n_m$ are distinct. We are choosing four distinct elements from the set (9) with 6 elements. Hence among these 4 chosen elements there exist two — say $\exp \frac{k_1 \pi i}{3}$, $\exp \frac{k_2 \pi i}{3}$, such that $|k_1 - k_2| = 3$, $k_1, k_2 \in \{1, \dots, 6\}$. Let $k_2 = k_1 + 3$, then

$$\exp \frac{k_2 \pi i}{3} = \exp \frac{k_1 \pi i}{3} \cdot \exp \pi i = - \exp \frac{k_1 \pi i}{3} ,$$

therefore there exist such numbers — say η_1, η_2 — that $\eta_2 = - \eta_1$ and

$$\exp \frac{2\pi i a_2}{n_m} = - \exp \frac{2\pi i a_1}{n_m} = \exp \left(\frac{2\pi i a_1}{n_m} + (2b + 1)\pi i \right) ,$$

hence $a_2 \equiv a_1 + \frac{n_m}{2} \pmod{n_m}$. Distinquish two cases.

a) If $v_1 = v_2$, then replacing $a_1(n_m)$, $a_2(n_m)$ by a single congruence $a_1 \left(\frac{n_m}{2} \right)$ we get a reduced system (see Remark 2) with one couple of equal moduli. According to Theorem 1 the moduli n_m and $\frac{n_m}{2}$ are of the from 2^c.

b) If $v_1 \neq v_2$, then we unite the first two congruences so that instead of them we obtain

either $a_1 \left(\frac{n_m}{2} \right)$, $a_1(n_m)$ with the coefficients v_2, $v_1 - v_2$ if $v_1 > v_2$

or $a_1 \left(\frac{n_m}{2} \right)$, $a_2(n_m)$ with the coefficients v_1, $v_2 - v_1$ if $v_2 > v_1$.

We have 3 equal moduli. Due to Theorem 2 the moduli n_m and $\frac{n_m}{2}$ are of the form $3^a \cdot 2^b$, $a \in \{0, 1\}$. Thus for $m = 4$ Theorem 4 is true

5. General step of induction

Suppose that $m > 4$ and the assertion of Theorem 4 is true for all systems with less then m congruences.

5.1.

Let (1) be a VCS in which there exist one triple $a_{m-2}(n_m)$, $a_{m-1}(n_m)$, $a_m(n_m)$ and on one couple $a_{t1}(n_0)$, $a_{t2}(n_0)$ of congruences with equal moduli $0 \leq a_{m-2}$, a_{m-1}, $a_m < n_m$, $0 \leq a_{t1}$, $a_{t2} < n_0$, $n_0 \in \{n_1, \dots, n_{m-3}\}$. Owing to Theorem 3 we obtain

$$\sum_{j=m-2}^{m} v_j \exp \frac{2\pi i a_j}{n_m} = 0 .$$

Using similar considerations as in part 2.2.2. we get three possibilities.

I. The vectors corresponding to $\exp \frac{2\pi i a_j}{n_m}$ are parallel. The impossibility of this case can be proved as in part 2.2.2.I.

II. The vectors corresponsing to $\exp \frac{2\pi i a_j}{n_m}$, $j = m-2, m-1, m$ form edges of a directed equilateral triangle. Similarly as in part 2.2.2. II. replacing the three congruences with equal moduli by a single congruence $a_{m-2}\left(\frac{n_m}{3}\right)$ we get a VCS with 2 or 3 equal moduli or with 2 couples of equal moduli. If this system is reduced then owing to Theorem 1 or Theorem 2 the assertion of Theorem 4 follows for this case. Suppose that the new system in which there are 2 couples of equal moduli is not reduced. Hence the new congruence $a_{m-2}\left(\frac{n_m}{3}\right)$ is contained in a MVS. Denote by M the set of remaining congruences of MVS. Due to $v_{m-2} = v_{m-1} = v_m$, $a_{m-2}\left(\frac{n_m}{3}\right)$ can be rewritten in three original congruences. Thus $M \cup \{a_{m-2}(n_m), a_{m-1}(n_m), a_m(n_m)\}$ forms the vanishing subsystem in (1) and this is a contradiction. Therefore the new VCS with 2 couples of equal moduli is reduced and the assertion of Theorem 4 follows owing to Theorem 2.

III. The vectors corresponding to $\exp \frac{2\pi i a_j}{n_m}$, $j = m, m-1, m-2$, are edges of an equilateral triangle in which one edge is directed oppositely to the others. Let $v_{m-2} = v_{m-1} = -v_m$. By a similar considerations as in part 2.2.2. III the last three congruences can be replaced by a $a_{m-2}\left(\frac{n_m}{3}\right)$ and $a_m\left(\frac{n_m}{2}\right)$. In the case of 2 or 3 equal moduli we use Theorem 1 or 2.

a) In the case of 2 couples of equal moduli we distinquish two possibilities.

There exist $n_j \in \{n_1, \ldots, n_{m-3}\} - \{n_0\}$ so that $n_j = \frac{n_m}{2}$ and $n_k \neq \frac{n_m}{3}$ for all k. If the new VCS is reduced, then the assortion of Theorem 4 follows according to Theorem 2. If the new VCS contains a vanishing subsystem, then it contain a MVS and $a_m\left(\frac{n_m}{2}\right)$, $a_j(n_j)$ belongs to this MVS.

Now if the congruence $a_{m-2}\left(\frac{n_m}{3}\right)$ belongs to the remaining reduced system then according to Theorem 1 its moduli are of the form 2^b and $\frac{n_m}{3} = 2^b$. Owing to Lemma 3 all the moduli in the MVS divide the largest modulus in it which is $\frac{n_m}{2} = 3 \cdot 2^{b-1}$.

Now if the congruence $a_{m-2}\left(\frac{n_m}{3}\right)$ belongs to the MVS, then we rewrite both congruences $a_{m-2}\left(\frac{n_m}{3}\right)$, $a_m\left(\frac{n_m}{2}\right)$ to the original congruences. Denote by M the set of the remaining congruences of MVS. Due to definition of a vanishing system for every integer r we have

$$0 = \sum_{t=m-2}^{m} v_t f_t(r) + v_m f_{m+1}(r) - v_m f_{m+1}(r) + \sum_{a_t(n_t) \in M} v_t f_t(r)$$

and hence

$$0 = \sum_{t=m-2}^{m} v_t f_t(r) + \sum_{a_t(n_t) \in M} v_t f_t(r) \tag{10}$$

where $f_{m+1}(r)$ is the characteristic function of the set $a_m + \frac{n_m}{2}(n_m)$ on Z. Thus the original system contains a vanishing system, which is a contradiction.

b) Further we can get a system with 3 couples of equal moduli in the case if there exist $n_{j1}, n_{j2} \in \{n_1, \ldots, n_{m-2}\} - \{n_0\}$ such that $n_{j1} = \frac{n_m}{3}$, $n_{j2} = \frac{n_m}{2}$. We have

$m - 1$ congruences here. If this system is reduced the assertion follows owing to the induction hypothesis. In the opposite case we distenqish the case when to the MVS belongs the first or the second or both new congruences (in this case (10) holds and a contradiction follows). We can get a MVS with one or two couples of equal moduli. Similarly as in parts a. and b. we use Theorem 1, 2, Lemma 3 and Lemma 4 as well. The moduli in the MVS will be of the required form.

c) Further we can get a system with one triple and one couple of equal moduli in the case if

$$\text{either } n_0 = \frac{n_m}{2} \text{ and there exists } n_j \in \{n_1, \dots, n_{m-3}\} - \{n_0\} \text{ so that } n_j = \frac{n_m}{3}$$

$$\text{or } n_0 = \frac{n_m}{3} \text{ and there exists } n_j \in \{n_1, \dots, n_{m-3}\} - \{n_0\} \text{ so that } n_j = \frac{n_m}{2}.$$

We shall analyse this second case (the first case is similar). If the new VCS is reduced then we have $m - 1$ congruences and the assertion of Theorem 4 follows from to induction hypothesis. Suppose that it is not reduced, i.e. it containd the MVS. If both new congruences belong to the MVS then (10) holds and a contradiction follows.

Suppose that only $a_m\left(\frac{n_m}{2}\right)$ belongs to the MVS (due to Lemma 2 the congruence $a_j(n_j)$ does too) $a_{m-2}\left(\frac{n_m}{2}\right)$ belongs to the remaining reduced system. Owing to Theorem 1 or 2 all moduli in the reduced system are of the form $3^a \cdot 2^b$, $a \in \{0, 1\}$. Thus $\frac{n_m}{3} = 3^a \cdot 2^b$. Due to Lemma 3 all moduli in the MVS divide $\frac{n_m}{2} = 3^{a+1} \cdot 2^{b-1}$.

Now suppose that only $a_{m-2}\left(\frac{n_m}{3}\right)$ belongs to the MVS and $a_m\left(\frac{n_m}{2}\right)$, $a_j(n_j)$ belong to the reduced system. According to Theorem 1 all moduli in the reduced system are of the form 2^b, $n_m = 2^b$ thus $\frac{n_m}{3} = \frac{2^{b+1}}{3}$ which is imposible. Hence a vanishing system does not exist in this case.

5.2

Let (1) be a VCS in whic there exist one couples $a_{m-1}(n_m)$, $a_m(n_m)$ and one triple $a_{t1}(n_0)$, $a_{t2}(n_0)$, $a_{t3}(n_0)$ of congruences with equal moduli. According to Theorem 3 we get (4) and $|v_{m-1}| = |v_m|$. If $v_{m-1} = -v_m$, then we get a contradiction as in part 2.2.1. If $v_{m-1} = v_m$, then the last two congruences can be replaced by a single congruence $a_{m-1}\left(\frac{n_m}{2}\right)$ as in part 2.2.1. If obtain a new system with one triple of equal moduli, then we use Theorem 2.

Now suppose that we have a new system with 4 or with one triple and one couple of equal moduli. The assertion follows from the induction hypothesis for the reduced system. Otherwise we shall proceed as in part 5.1. II. and we get a contradiction.

5.3.

Le (1) be a VCS in which there exist 4 equal moduli: $2 \leq n_1 < \ldots < n_{m-4} < < n_{m-3} = \ldots = n_m$. Due to Thorem 3 we obtain

$$\sum_{j=m-3}^{m} v_j \exp \frac{2\pi i a_j}{n_m} = 0 .$$

Using similar considerations as in part 4. we can show the existence of $\exp \frac{2\pi i a_j}{n_m}$, $\exp \frac{2\pi i a_k}{n_m}$ such that

$$\exp \frac{2\pi i a_j}{n_m} = -\exp \frac{2\pi i a_k}{n_m}. \text{ Then } a_j \equiv a_k + \frac{n_m}{2} (\operatorname{mod} n_m), \text{ too.}$$

Let $k = m - 1, j = m$.

a) If $v_{m-1} = v_m$, then we get a new system with one or two couples of equal moduli. If it is reduced, then the assertion follows according to Theorem 1 or 2. If it is not reduced, then $a_j\left(\frac{n_m}{2}\right)$ bolongs to the MVS and the second couple of congruences with equal moduli n_m belongs to the remaining reduced system where the moduli are of the form 2^b and $n_m = 2^b$ follows from Theorem 1. Owing to the Lemma 3 all moduli in the MVS divide $\frac{n_m}{2} = 2^{b-1}$. Thus all moduli in (1) are of the required form.

b) If $v_m \neq v_{m-1}$ then replace $a_m(n_m)$, $a_{m-1}(n_m)$

either by $a_m\left(\frac{n_m}{2}\right)$, $a_m(n_m)$ with the coefficients v_{m-1}, $v_m - v_{m-1}$ if $v_m > v_{m-1}$

or by $a_m\left(\frac{n_m}{2}\right)$, $a_{m-1}(n_m)$ with the coefficients v_m, $v_{m-1} - v_m$ if $v_m < v_{m-1}$.

If there is one triple of equal moduli in the new VCS then the assertion follows from Theorem 2. If we get a VCS with one triple and one couple of equal moduli then the moduli are of the required form, according to 5.1., in the case of the reduced system.

Now suppose that the new system is not reduced. Then $a_m\left(\frac{n_m}{2}\right)$ belongs to the MVS. Owing to Theorem 2 all moduli in the remaining reduced system are of the form $3^a \cdot 2^b$ and $n_m = 3^a \cdot 2^b$. Due to Lemma 3 all moduli in the MVS divide $\frac{n_m}{2} = 3^a \cdot 2^{b-1}$.

5.4.

Let (1) be a VCS in which there exist three couples of equal moduli. Due to Theorem 3 we obtain (4) and then $|v_{m-1}| = |v_m|$ holds. If $v_{m-1} = -v_m$, then we get a contradiction as in part 2.2.1. If $v_{m-1} = v_m$, then the last two congruences can be replaced by single congruence $a_{m-1}\left(\frac{n_m}{2}\right)$ as in part 2.2.1. We get a system with 2 or 3 equal moduli or with one triple and one couple of equal moduli. If it is reduced system then the moduli are of the required form according to Theorem 2 or the induction hypothesis. If it is not a reduced system then we get a contradiction as in part 5.1. II. Thus a vanishing system does not exist in this case.

The proof of Theorem 4 is complete.

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РЕЗЮМЕ

ВЕКТОРНО-ПОКРЫВАЮЩИЕ СИСТЕМЫ С ЧЕМЫРЬМЯ ОДИНАКОВЫМИ МОДУЛЯМИ

Ивета Данешова, Братислава

В данной статье рассматрываются свойства некоторых типов точно и векторно-покрывающих систем. Новым результатом является теорема об векторно-покрывающей системе с четырьмя одинаковыми модулями. Все модули вида $3^a \cdot 2^b$, $a \in \{0, 1, 2\}$, $b \in Z$, $b \geqslant 0$.

SÚHRN

VEKTOROVO-POKRÝVAJÚCE SÚSTAVY SO ŠTYRMI ROVNAKÝMI MODULMI

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V článku uvádzame niektoré vlastnosti presne a vektorovo-pokrývajúcich sústav. Novým výsledkom je veta o vektorovo-pokrývajúcej sústave so štyrmi rovnakými modulmi. Moduly v tejto sústave majú tvar $3^a \cdot 2^b$, $a \in \{0, 1, 2\}$, b je nejaké celé nezáporné číslo.