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ASYMPATOTICAL DENSITIES IN GENERALIZED PASCAL TRIANGLES (EXTENDED ABSTRACT)[1])

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Generalized Pascal triangles (GPT) were defined in [4] in connection with the study of structure of regular systolic trellis automata, see [1]—[3]. GPT are constructed similarly as the usual Pascal triangle is, only operations of a finite algebra $\mathscr{A}$ are used instead of the addition of integers (the exact definition will follow). For example, the Pascal triangle modulo d (d is a positive integer) is a special case of GPT. Singmaster [7] proved that for every positive integer d almost all binomial coefficients are multiples of d. From that it follows that the (two-dimensional) asymptotical density of 0 in the Pascal triangle modulo d is equal to 1, and the asymptotical densities of the other elements are equal to 0.

Here we present (without proofs) some results about asymptotical densities in GPT. Roughly speaking, these densities can be almost arbitrary. Only some recursivity conditions and obvious inequalities must be fulfilled.

All necessary notions and notations we use are from [4]. Here we only repeat the definition of GPT and briefly outline some other notions. Z, N will denote the set of integers, or nonnegative integers, respectively. "Algebra" will always mean a finite algebra $\mathscr{A} = (\mathrm{A}; K, \mathrm{l}, \mathrm{r},.)$ of signature (0, 1, 1, 2).

Definition 1.

a) To every algebra $\mathscr{A} = (\mathrm{A}; K, \mathrm{l}, \mathrm{r},.)$ and every word $w = a_0a_1 \dots a_n$ ($n \geqq 0$ and $a_i \in \mathrm{A}$ for all $i = 0, \dots, n$) we associate the function $G = \mathrm{GPT}(\mathscr{A}, w)$ with the domain

$$\mathrm{D}_{0,n} = \{(x, y) \in \mathrm{Z} \times \mathrm{Z}, \quad x \geqq 0, \quad x + y \geqq 0 \quad \text{and} \quad y \geqq -n\}$$

by the formulae

$$\begin{array}{lll} G(x, -x) & = a_x & \text{for } x = 0, \dots, n \\ G(0, y+1) & = \mathrm{l}(G(0, y)) & \text{for } y \geqq 0 \\ G(x+1, -n) & = \mathrm{r}(G(x, -n)) & \text{for } x \geqq n \\ G(x+1, y+1) & = G(x, y+1) \cdot G(x+1, a) & \text{for} x \geqq 0, y \geqq -n, x+y+2 > 0. \end{array}$$

[1]) The full version has appeared in Computers and Artificial Intelligence.

ALGEBRA A:

.	R	M	1	2
L	1	M	1	2
M	M	M	1	2
1	1	1	2	M
2	2	2	M	1

GPT(A, 222):

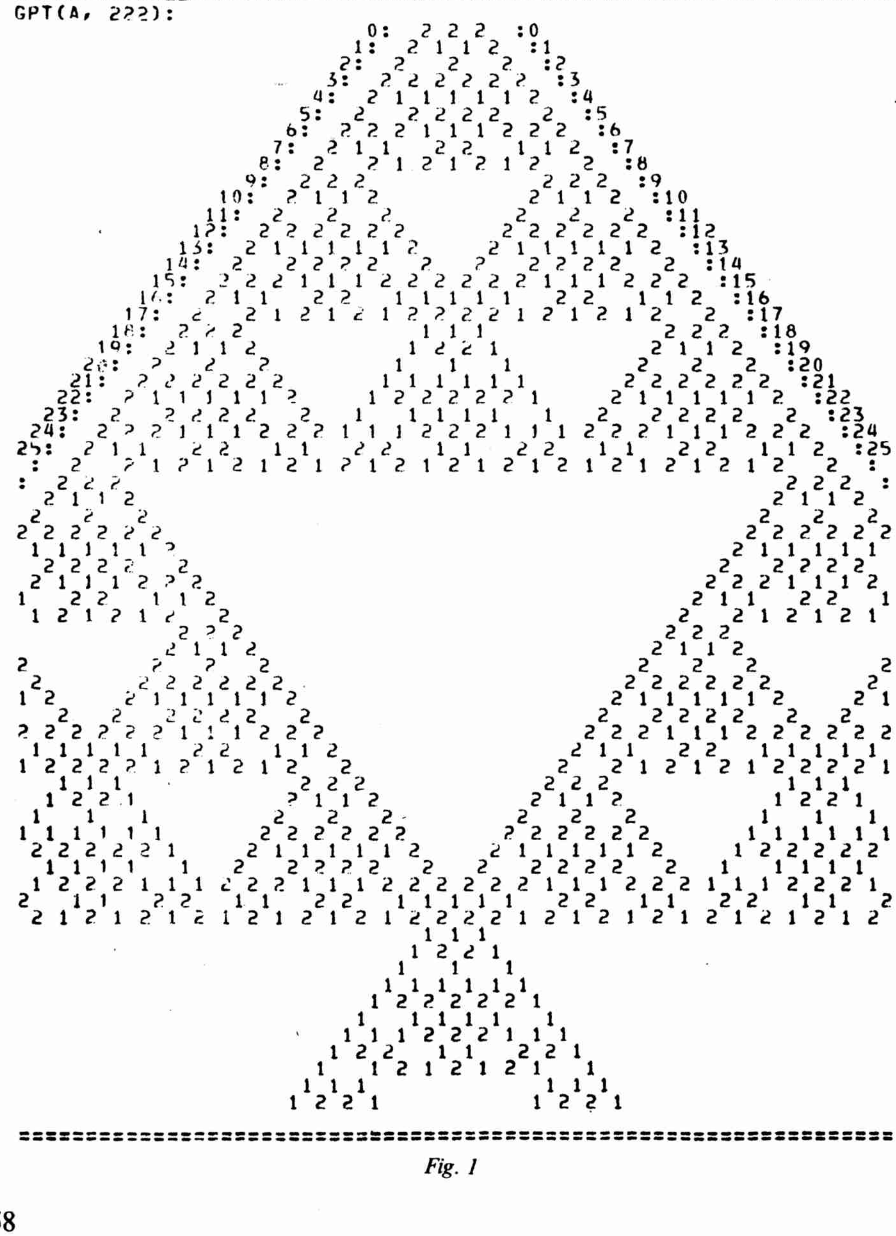

Fig. 1

ALGEBRA A:

.	R	M	1	2
L	1	1	2	1
M	2	M	1	2
1	1	1	2	M
2	2	2	M	1

GPT(A):

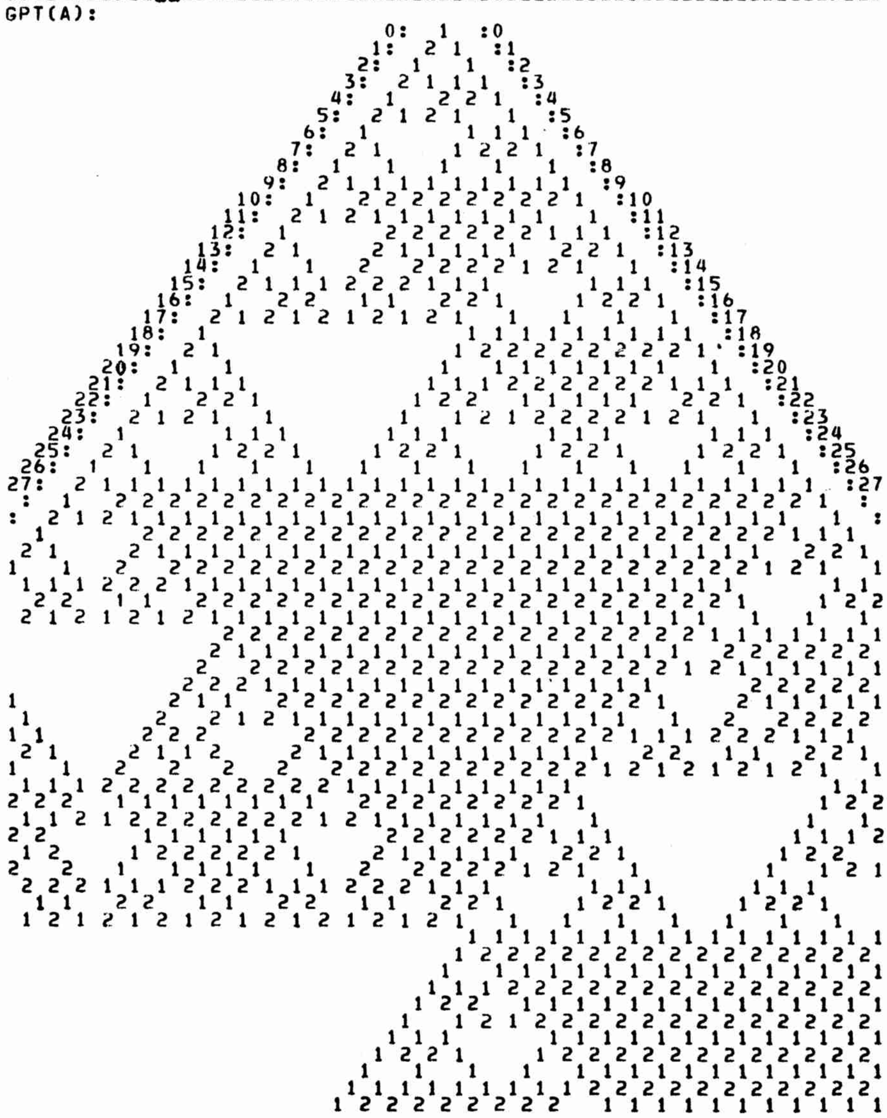

Fig. 2

The function G is called the GPT associated to the algebra $\mathscr{A}$ and the word w.

b) We denote $\mathrm{GPT}(\mathscr{A}) = \mathrm{GPT}(\mathscr{A}, K)$; this function is called the GPT associated to $\mathscr{A}$.

Hence the classical Pascal triangle is associated to the algebra $\mathscr{N} = (\mathrm{N}; 1, \mathrm{id}_{\mathrm{N}}, \mathrm{id}_{\mathrm{N}}, +)$. We draw all GPT analogously as the classical Pascal triangle, and so we can speak about their rows, columns, margins, and diagonals in the obvious way. (Left and right diagonals are parallel with the left and right margins, respectively.) Notice that every column has nonempty intersection either with (almost all) even rows or with odd rows.

In the next definition the following notation will be useful

$$\mathrm{D}^n_{0,k} = \{(i, j) \in \mathrm{D}_{0,k};\quad i + j < n\}.$$

Informally, $\mathrm{D}^n_{0,k}$ consists of (the elements of) the first n lines of the set $\mathrm{D}_{0,k}$.

Deffinition 2. For every algebra $\mathscr{A}$, every word $w \in \mathrm{A}^+$ and every element $x \in \mathrm{A}$ the number

$$\mathrm{Dens}(\mathscr{A}, w, x) = \lim_{n \to \infty} \frac{\mathrm{card}(\{(i, j) \in \mathrm{D}^n_{0,|w|-1};\ G(i, j) = x\})}{\mathrm{card}(\mathrm{D}^n_{0,|w|-1})}$$

is called the asymptotical density of x in $G = \mathrm{GPT}(\mathscr{A}, w)$. The lower density $\mathrm{Densinf}(\mathscr{A}, w, x)$ and the upper density $\mathrm{Denssup}(\mathscr{A}, w, x)$ are defined analogously (lim is replaced by lim inf, lim sup, respectively). Further, we shall write $\mathrm{Dens}(\mathscr{A}, x)$ instead of $\mathrm{Dens}(\mathscr{A}, K, x)$, and analogously for Densinf, Denssup.

$\mathrm{Dens}(\mathscr{A}, w, x)$ need not exist but the lower and the upper density of x in $\mathrm{GPT}(\mathscr{A}, w)$ always exists and it holds

$$0 \leqq \mathrm{Densinf}(\mathscr{A}, e, x) \leqq \mathrm{Denssup}(\mathscr{A}, w, x) \leqq 1\,.$$

$\mathrm{Dens}(\mathscr{A}, w, x)$ exists if and only if the equality holds in the centre of the above formula.

We begin with two theorems about rational asymptotical densities.

Theorem 1. For every ordered $(s + 1)$-tuple

$$\alpha_0, \alpha_1, \ldots, \alpha_s \tag{1.1}$$

of nonnegative rational numbers such that

$$\alpha_0 + \alpha_1 + \ldots + \alpha_s = 1 \tag{1.2}$$

there is a finite algebra $\mathscr{A}$ such that $\{0, 1, \ldots, s\} \subseteq \mathrm{A}$ and

$$\mathrm{Dens}(\mathscr{A}, i) = \alpha_i \tag{1.3}$$

for all $i = 0, 1, \ldots s$.

Theorem 2. For every $s \in \mathrm{N}$ there is a finite algebra $\mathscr{A}$ such that $\{0, 1, \ldots, s\} \subseteq \mathrm{A}$ and for every ordered $(s + 1)$-tuple (1.1) of nonnegative rational numbers such that

ALGEBRA A:

.	R	M	1	2	3	4
L	1	M	1	2	3	4
M	M	M	1	2	3	4
1	1	1	2	3	4	M
2	2	2	3	4	M	1
3	3	3	4	M	1	2
4	4	4	M	1	2	3

GPT(A):

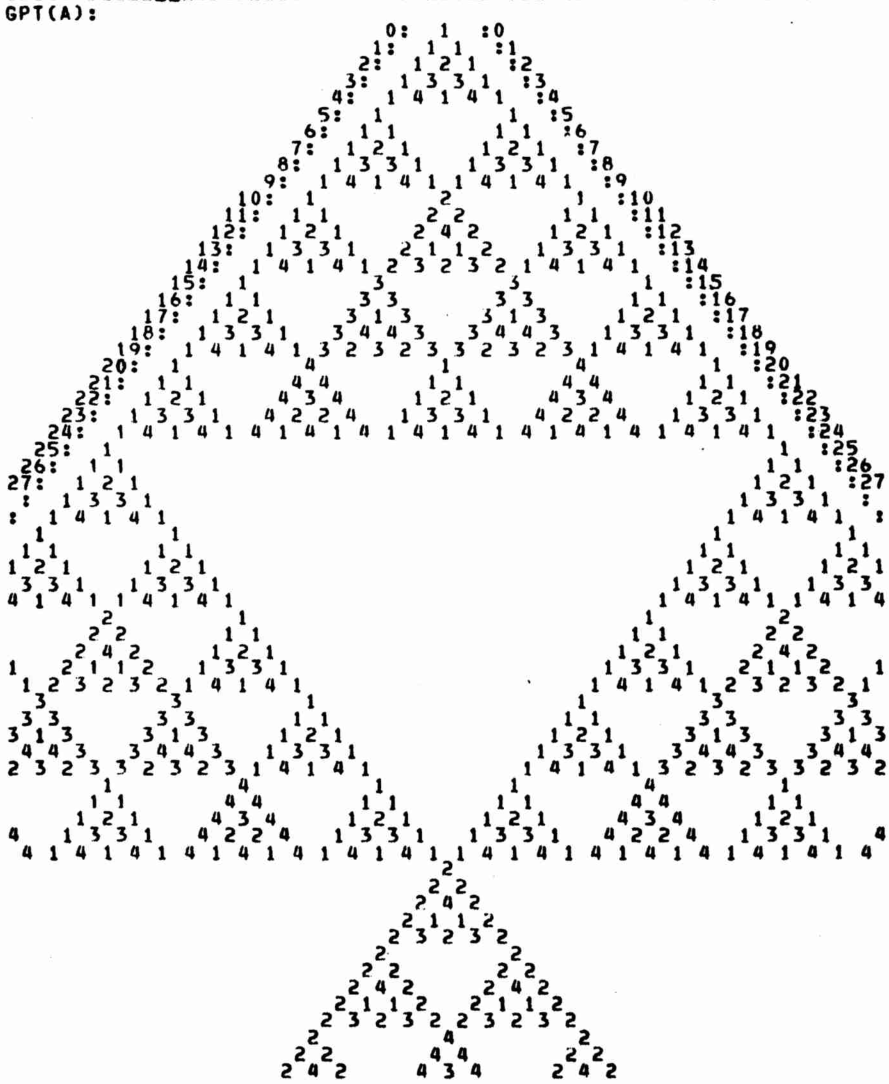

Fig. 3

(2.2) $$\alpha_0 + \alpha_1 + \ldots + \alpha_s \leqq 1$$

there is $w \in A^+$ such that

(2.3) $$\mathrm{Dens}(\mathscr{A}, w, i) = \alpha_i$$

for all $i = 0, 1, \ldots, s$.

The main advantage of Theorem 2 is that we have unique algebra for all $(s+1)$-tuples of rational (s fixed); the densities are only controlled by the initial word w. (The difference between (1.2) and (2.2) is less substantial because we can join $\alpha_{s+1} = 1 - (\alpha_0 + \ldots + \alpha_s)$ to (1.1) if necessary.) Further theorems will be similar to Theorem 2 in this direction; the appropriate analogons of Theorem 1 also hold.

Definition 3. A real α will be called limiting recursive if there is a recursive sequence of rational numbers with the limit α.

No recursive bound for the speed of convergence is requested above; it is requested in the definition of recursive reals. There are limiting recursive reals which are not recursive, see e.g. [5] or [6].

Theorem 3. For every $s \in \mathrm{N}$ there is a finite algebra $\mathscr{A}$ such that $\{0, 1, \ldots, s\} \subseteq \mathrm{A}$ and for every $(s+1)$-tuple

(3.1) $$\alpha_0, \alpha_1, \ldots, \alpha_s$$

of nonnegative limiting recursive reals satisfying

(3.2) $$\alpha_0 + \alpha_1 + \ldots + \alpha_s \leqq 1$$

there is a word $w \in A^+$ such that

(3.3) $$\mathrm{Dens}(\mathscr{A}, w, i) = \alpha_i$$

for $i = 0, 1, \ldots, s$.

While the first two theorems can be proved rather directly, the proof of Theorem 3 uses the idea of simulation of a universal Turing machine $\mathscr{T}$. Roughly speaking, the machine $\mathscr{T}$ computes approximations of the reals (3.1) which are determined by the word w.

Theorem 4. For every $s \in \mathrm{N}$ there is a finite algebra $\mathscr{A}$ such that $\{0, 1, \ldots, s\} \subseteq \mathrm{A}$ and for every ordered $(2s+2)$-tuple of nonnegative limiting recursive reals

(4.1) $$\alpha_0, \ldots, \alpha_s, \quad \beta_0, \ldots, \beta_s$$

which satisfy the conditions

(4.2) $$\alpha_i \leqq \beta_i$$

(4.3) $$\alpha_0 + \ldots + \alpha_{i-1} + \beta_i + \alpha_{i+1} + \ldots + \alpha_s \leqq 1$$

for all $i = 1, \ldots, s$, there is a word $w \in A^+$ such that

(4.4) $$\mathrm{Densinf}(\mathscr{A}, w, i) = \alpha_i \text{ and } \mathrm{Denssup}(\mathscr{A}, w, i) = \beta_i$$

ALGEBRA A:

.	R	M	1	2	3	4	5
L	1	.	1	.	.	.	.
M	.	M	1	M	4	M	M
1	1	1	2	3	2	.	.
2	.	M	3	1	.	5	.
3	.	4	2	.	4	.	M
4	.	M	.	5	.	1	.
5	.	M	.	.	M	.	3

GPT(A):

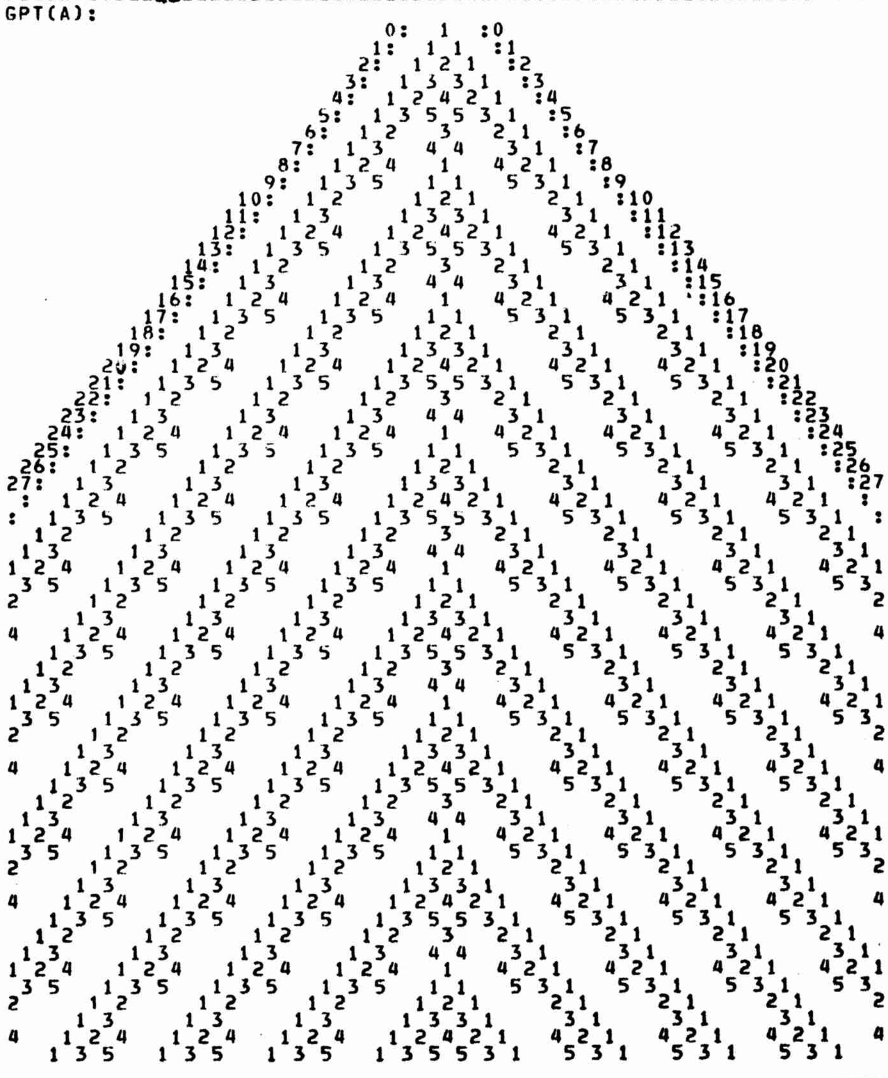

Fig. 4

for all $i = 0, 1, s$. Moreover, if for all $i = 0, 1, \ldots, s$ we have

$$\beta_0 + \ldots + \beta_{i+1} + \alpha_i + \beta_{i+1} + \ldots + \beta_s \geqq 1, \tag{4.5}$$

then we could arrange $\mathrm{Dens}(\mathscr{A}, w, x) = 0$ for all

$x \in A - \{0, 1, \ldots, s\}$.

The proof of Theorem 4 also uses a simulation of the universal Turing machine $\mathscr{T}$. To obtain suitable approximations of the reals (4.1), the machine $\mathscr{T}$ computes the function F from the next lemma.

Lemma. For every $(2s + 2)$-tuple (4.1) of nonnegative limiting recursive reals satisfying the conditions (4.2), (4.3), and (4.5) for all $i = 0, 1, \ldots, s$ there is a binary recursive function $F(x, y)$ such that

$$F(i, j) < F(i, k) \quad \text{for all} \quad i, j, k \in \mathrm{N}, j < k$$

and

$$\liminf_{k \to \infty} \frac{F(i, k)}{F(0, k) + \ldots + F(s, k) + s + 1} = \alpha_i \tag{4.6}$$

$$\limsup_{k \to \infty} \frac{F(i, k)}{F(0, k) + \ldots + F(a, k) + s + 1} = \beta_i \tag{4.7}$$

for all $i = 0, 1, \ldots, s$.

Figures display several GPT. The upper part of each of them contains a Cayley table of an algebra $\mathscr{A}$. Dots in the table (except that in the upper left corner) denote that the corresponding values can be arbitrary. The lower part displays the corresponding GPT ($\mathscr{A}$) or GPT ($\mathscr{A}$, w); the rows are numbered till possible. To obtain nicer figures, the element M is replaced by the blank. The figures are understable without further expolantion.

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SÚHRN

ASYMPTOTICKÉ HUSTOTY V ZOVŠEOBECNENÝCH PASCALOVÝCH TROJUHOLNÍKOCH

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Vyšetrujú sa dolné a horné dvojrozmerné asymptotické hustoty prvkov v zovšeobecnených Pascalových trojuholníkoch konečných algebier. Dokazuje sa, že tieto hustoty môžu byť takmer ľubovoľné, až na niektoré podmienky rekurzívnosti a zrejmé nerovnosti.

РЕЗЮМЕ

АСИМПТОТИЧЕСКИЕ ПЛОТНОСТИ В ОБОБЩЕННЫХ ТРЕУГОЛЬНИКАХ ПАСКАЛЯ

Иван Корец, Братислава

Исследуются верхние и нижние асимптотические плотности элементов в обобщенных треугольниках Паскаля для конечных алгебр. Доказывается, что они почти произвольны. Только некоторые условия рекурсивности и очевидные неравенства должны иметь метсо.