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# CONVERGENCE OF SUBSERIES OF THE HARMONIC SERIES AND ASYMPTOTIC DENSITIES OF SETS OF INTEGERS (PRELIMINARY ANNOUNCEMENT)

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There exists a relation between the convergence of subseries

$$\sum_{n=1}^{\infty} k_n^{-1} \tag{1}$$

of the harmonic series $\sum_{n=1}^{\infty} n^{-1}$ and asymptotic densities of the sets $K = \{k_1 < k_2 < \dots < k_n < \dots\}$ (see Theorem A). We shall show that this relation cannot be essentially improved.

If $M \subset N = \{1, 2, \dots, n, \dots\}$, then we put

$$M(x) = \sum_{a \in M, a \leqq x} 1$$

and

$$d(M) = \lim_{x \to \infty} \frac{M(x)}{x}$$

if the limit on the right-hand side exists. The number $d(M)$ is called the asymptotic density of the set $M$ (cf. [1], p. xix).

**Theorem A.** If $\sum_{n=1}^{\infty} k_n^{-1} < +\infty$, then $d(K) = 0$.

For the proof of Theorem A see e.g. [4], Theorem 1. Let us remark that the theorem A can be easily deduced also from the folowing result:

Let $\sum_{n=1}^{\infty} a_n$ be a series with real terms, let

$$a_1 \geqq a_2 \geqq \dots \geqq a_n \geqq a_{n+1} \geqq , \quad a_n \to 0, \sum_{n=1}^{\infty} a_n = +\infty .$$

Denote by $N(x)$ the number of $n'$ s with $a_n \geqq x > 0$. Then we have

(2) $$\lim_{x \to 0+} x N(x) = 0$$

(cf. [3], [5]). If we put $a_n = k_n^{-1}$ $(n = 1, 2, \ldots)$, then we have

$$N(x) = \# \{n\colon a_n \geqq x\} = \# \{n\colon k_n \leqq x^{-1}\} = K(x^{-1})$$

and hence

$$0 = \lim_{x \to 0+} x N(x) = \lim_{x \to 0+} \frac{K(x^{-1})}{x^{-1}} = \lim_{y \to \infty} \frac{K(y)}{y} = d(K)$$

In the paper [2] the following theorem is introduced (see Theorem 3 from [2]):

**Theorem B.** If $M \subset N$ and $\sum_{m \in M} m^{-1} < +\infty$, then $c_M = 0$, where

$$c_M = \lim_{x \to \infty} M(x) \frac{\log x}{x}$$

(the author of [2] uses the notation $\nu_M(x)$ instead of $M(x)$).

The following two examples show that the theorem B does not hold.

**Example 1.** Put $M = \bigcup_{n=2}^{\infty} M_n$, where

$$M_n = \{n^{n^2} + 1, n^{n^2} + 2, \ldots, n^{n^2} + n^{n^2-2}\} \quad (n = 2, 3, \ldots)$$

Then a simple estimation yields

$$\sum_{m \in M_n} m^{-1} \leqq n^{n^2-2} \frac{1}{n^{n^2}} = \frac{1}{n^2} \quad (n = 2, 3, \ldots),$$

Hence $\sum_{m \in M} m^{-1} < +\infty$.

Putting $y_n = n^{n^2} + n^{n^2-2}$ we get

$$M(y_n) \geqq n^{n^2-2}, \ \log y_n \geqq n^2 \log n$$

hence

$$M(y_n) \frac{\log y_n}{y_n} \geqq n^{n^2-2} \frac{n^2 \log n}{2n^{n^2}} = \frac{\log n}{2} \to +\infty$$

(as $n \to \infty$). Therefore we have

$$\limsup_{x \to \infty} M(x) \frac{\log x}{x} = +\infty$$

**Example 2.** Theorem B is not valid even in the case of subseries of the series $\Sigma p^{-1}$ (the reciprocals of all prime numbers).

Let $P = \{p_1 < p_2 < \dots < p_n < \dots\}$ be the set of all prime numbers. Put $Q = \bigcup_{n=2}^{\infty} Q_n$, where (writting $p(k)$ instead of $p_k$)

$$Q_n = \{p(n^{n^2} + 1), p(n^{n^2} + 2), \dots, p(n^{n^2} + t_n)\},$$
$$t_n = [n^{-2} p(n^{n^2})] \quad (n = 1, 2, \dots).$$

A simple estimation yields $\sum_{q \in Q} q^{-1} \leqq \sum_{n=1}^{\infty} n^{-2} < +\infty$.

Put

$$A_n = Q(y_n) \frac{\log y_n}{y_n} \quad (n = 1, 2, \dots),$$

where

$$y_n = p(n^{n^2} + t_n) \quad (n = 2, 3, \dots).$$

We shall use the well-known Tchebysheff's inequalities

$$a n \log n < p_n < b n \log n \quad (n = 1, 2, \dots),$$

where $a$, $b$ are positive constants.

We have

$$Q(y_n) \geqq t_n > \frac{1}{2} a\, n^{n^2} \log n \quad (\text{for } n > n_0) \tag{3}$$

Further

$$y_n > a n^{n^2} \cdot n^2 \log n,$$

hence

$$\log y_n > n^2 \log n + O(\log n) \quad (n \geqq 2) \tag{4}$$

By a simple estimation we obtain

$$n^{n^2} + t_n \leqq n^{n^2}(1 + b \log n)$$

Hence

$$y_n \leqq b(n^{n^2} + t_n) \log (n^{n^2} + t_n) \leqq$$
$$\leqq b n^{n^2}(1 + b \log n) \log \{n^{n^2}(1 + b \log n)\} =$$
$$= b^2 n^{n^2+2} \cdot \log^2 n + O(n^{n^2+2} \log n)$$

Thus we get

$$y_n \leqq b^2 n^{n^2+2} \log^2 n + O(n^{n^2+2} \cdot \log n) \tag{5}$$

From (3)—(5) we get (for $n > n_0$)

$$A_n \geqq \frac{a}{2b^2} \frac{n^{n^2} \log n(n^2 \log n + O(\log n))}{n^{n^2+2} \cdot \log^2 n + O(n^{n^2+2} \cdot \log n)} = \frac{a}{2b^2} \frac{\log^2 n + o(1)}{\log^2 n + O(\log n)}$$

Fromt this it is evident that

$$\limsup_{x \to \infty} Q(x) \frac{\log x}{x} \geqq \frac{a}{2b^2} > 0$$

**Remark 1.** Let us remember that in [4] the following result is proved (Theorem 2 in [4]):

Let $d_1 \geqq d_2 \geqq \ldots, \sum_{n=1}^{\infty} d_n = +\infty$ and $\sum_{k=1}^{\infty} \varepsilon_k(x) d_k < +\infty$, where $\varepsilon_k(x)(k = 1, 2, \ldots)$ are dyadic digits of the number $x \in (0, 1\rangle$ $\left(\text{i.e. } x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k}\right.$ is the non-terminating dyadic expansion of $\left.x\right)$.

Then

$$p_1(x) = \liminf_{n \to \infty} \frac{p(n, x)}{n} = 0$$

where $p(n, x) = \sum_{k=1}^{n} \varepsilon_k(x)$ $(n = 1, 2, \ldots)$.

If we apply this result to subseries of the series $\sum_{n=1}^{\infty} p_n^{-1}$ we see that the convergence of such subseries implies that "the lower density" of this subseries in $\sum_{n=1}^{\infty} p_n^{-1}$ is zero.

The foregoing examples suggest the formulation and the proof of the following theorem which shows that the result obtained in Theorem A cannot be essentially improved. The proof of the following theorem will be published in a forthcoming paper together with the proofs of some further results.

**Theorem 1.** Let $g$: $(0, +\infty) \to (0, +\infty)$, let

$$\lim_{x \to \infty} g(x) = +\infty \tag{6}$$

(arbitrarily slowly). Then there exists an infinite set $M \subset N$ such that $\sum_{m \in M} m^{-1} < +\infty$ and simultaneously

$$\limsup_{x \to \infty} M(x) \frac{g(x)}{x} = +\infty$$

The assertion of Theorem 1 remains true also in the case if (6) will be replaced by the weaker condition:

$$\limsup_{x \to \infty} g(x) = +\infty$$


## REFERENCES

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## SÚHRN

## KONVERGENCIA ČIASTOČNÝCH RADOV HARMONICKÉHO RADU A ASYMPTOTICKÉ HUSTOTY MNOŽÍN PRIRODZENÝCH ČÍSEL


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Nech $M = \{m_1 < m_2 < m_3 < \ldots\}$ je množina prirodzených čísel, pričom

$$\sum_{n=1}^{\infty} m_n^{-1} < +\infty \tag{1}$$

Potom, ako je známe, platí $d(M) = 0$, kde $d(M)$ označuje asymptotickú hustotu množiny $M$. V práci [2] je uvedené tvrdenie (pozri vetu 3 z [2]), podľa ktorého z (1) vyplýva

$$\lim_{x \to \infty} M(x) \frac{\log x}{x} = 0$$

kde $M(x) = \sum_{n: m_n \leq x} 1$. V tejto poznámke je na príkladoch ukázané, že uvedené tvrdenie je nepravdivé.

V súvislosti s predchádzajúcim sa v tomto článku uvádza nasledujúca veta (bez dôkazu):

Nech $\lim g(x) = +\infty$. Potom existuje taká množina $M = \{m_1 < m_2 < m_3 \ldots\}$, pre ktorú platí (1) a súčasne

$$\lim_{x \to \infty} M(x) \frac{g(x)}{x} = +\infty$$

## РЕЗЮМЕ

## СХОДИМОСТЬ ЧАСТЕЙ РЯДОВ ГАРМОНИЧЕСКОГО РЯДА И АСИМПТОТИЧЕСКИЕ ПЛОТНОСТИ МНОЖЕСТВ НАТУРАЛЬНЫХ ЧИСЕЛ

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Пусть $M = \{m_1 < m_2 < \ldots\}$ множество натуральных чисел и пусть

(1) $$\sum_{n=1}^{\infty} m_n^{-1} < +\infty$$

Потом, как известно, $d(M) = 0$, где $d(M)$ обозначает асимптотическую плотность множества $M$. В работе [2] (см. теоремы 3 из [2] утверждается что из (1) следует

$$\lim_{x \to \infty} M(x) \frac{\log x}{x} = 0,$$

где $M(x) = \sum_{n: m_n \leq x} 1$. В этой заметке доказано на примерах что это утверждение не верно. В связу с предидущим вводится здесь следующая теорема (без доказательства):

Пусть $\lim_{x \to \infty} g(x) = +\infty$. Тогда существует такое множество $M$ для которого выпольняется (1) и одновременно

$$\lim_{x \to \infty} \frac{M(x) g(x)}{x} = +\infty.$$