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ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
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SOME APPLICATIONS OF FRANEL'S INTEGRAL, I

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Introduction

In [1] J. Franel used the following integral

$$\int_0^1 \left(\{at\} - \frac{1}{2}\right)\left(\{bt\} - \frac{1}{2}\right) \mathrm{d}t = \frac{1}{12} \frac{(a, b)^2}{a \cdot b}, \tag{1}$$

where (a, b) denotes the greatest common divisor of positive integers a, b and $\{t\}$ is the fractional part of t.

In the theory of uniform distribution and particularly in the definition of the so-called L^2 discrepancy, the integral

$$\int_0^1 R_N^2(t)\,\mathrm{d}t \tag{2}$$

plays an important role. Here, by standard notations in [2],

$$R_N(t) = A([0, t); N) - Nt$$

$$A([0, t); N) = \mathrm{card}\{i;\ i \leqq N,\ 0 \leqq x_i < t\}$$

for a finite nondecreasing sequence

$$x_1 \leqq x_2 \leqq \ldots \leqq x_N \tag{3}$$

of real numbers from the interval [0, 1].

Some calculations of the integral (2) are very well known, for example (2) is equal to

$$N \sum_{i=1}^{N} \left(\frac{2i-1}{2N} - x_i\right)^2 + \frac{1}{12} \tag{4}$$

or, equivalently,

$$\left(\sum_{j=1}^{N}\left(\frac{1}{2} - x_j\right)\right)^2 + \frac{1}{2\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}\left|\sum_{j=1}^{N} \mathrm{e}^{2\pi i h x_j}\right|^2 \tag{5}$$

see [2, p. 161 and p. 110].

In Part I we shall calculate the integral (2) using the integral (1) for special sequences (3) of rational numbers. Also for these sequences we shall calculate (2) using (5) and the very well known Ramanujan identity (cf. [3, p. 197])

$$\sum_{\substack{0 < x < b \\ (x, b) = 1}} e^{2\pi i \frac{a}{b} x} = \frac{\varphi(b)}{\varphi\left(\frac{b}{(a, b)}\right)} \mu\left(\frac{b}{(a, b)}\right), \tag{6}$$

where φ, μ are standard Euler and Möbius number-theoretic functions.

In Part II we shall point at some connection between the summ of squares (4) and the Duffin—Schaefer conecture.

I.

Let $q_1, q_2, \ldots, q_n$ be a finite sequence of positive integers and

$$d_{ij} = (q_i, q_j), \quad q_{ij} = \frac{q_i q_j}{d_{ij}^2}$$

Let us order by magnitude the sequence of all reduced fractions from the interval [0, 1] whose denominators are from $q_1, q_2, \ldots, q_n$, i.e.

$$\frac{x(1)}{q_{i(1)}} \leqq \frac{x(2)}{q_{i(2)}} \leqq \ldots \leqq \frac{x(N)}{q_{i(N)}},$$

where all $q_{i(j)}$ are from $q_1, q_2, \ldots, q_n$ and*)

$$0 < x(j) \leqq q_{i(j)}, \quad (x(j), q_{i(j)}) = 1, \quad j = 1, 2, \ldots, N, \quad N = \sum_{i=1}^{n} \varphi(q_i). \tag{7}$$

Next, let us abbreviate

$$v(x) = \text{card}\{p;\ p|x,\ p \text{ is a prime number}\}$$

$$\text{ind}_p(x) = \max\{\alpha;\ p^\alpha|x,\ \alpha \text{ is an integer}\}.$$

Theorem 1. For every finite sequence $q_1, q_2, \ldots, q_n$ of integers greater than 1 and N, $x(j)/q_{i(j)}, j = 1, 2, \ldots, N$ which are defined by (7) it holds

$$N \cdot \sum_{j=1}^{N} \left(\frac{2j-1}{2N} - \frac{x(j)}{q_{i(j)}}\right)^2 + \frac{1}{12} = \tag{8}$$

*) How many times $q_{i(j)}$ is contained in $q_1, q_2, \ldots, q_n$ so often $x(j)/q_{i(j)}$ is contained in (7).

$$= \frac{1}{12} \sum_{i,j=1}^{n} \frac{2^{v(d_{ij})}}{q_{ij}} \prod_{\substack{p \backslash q_i q_j \\ p \nmid d_{ij}}} (1-p) \prod_{\substack{p \backslash d_{ij} \\ p \nmid q_{ij}}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \backslash d_{ij} \\ p \backslash q_{ij}}} \left(1 - \frac{p}{2}\left(1 + \frac{1}{p^2}\right)\right) \tag{9}$$

$$= \frac{1}{2\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| \sum_{i=1}^{n} \frac{\varphi(q_i)}{\varphi\left(\frac{q_i}{(h, q_i)}\right)} \mu\left(\frac{q_i}{(h, q_i)}\right) \right|^2 \tag{10}$$

Proof. Let us substitute to (3) the sequence (7). According to the elementary sieve we have

$$\sum_{i=1}^{n} \sum_{\substack{0 < \frac{x}{q_i} \leq t \\ (x, q_i) = 1}} 1 = \sum_{i=1}^{n} \left(t\varphi(q_i) - \sum_{d \backslash q_i} \mu(d) \left\{ \frac{q_i t}{d} \right\} \right) = A([0, t]; N).$$

Since the integral (2) is independent of values R_N in points x_i, applying (1) we have

$$\int_0^1 R_N^2(t)\,\mathrm{d}t = \int_0^1 \left(\sum_{i=1}^{n} \sum_{d \backslash q_i} \mu(d) \left\{ \frac{q_i t}{d} \right\} \right)^2 \mathrm{d}t$$

$$= \sum_{i,j=1}^{n} \sum_{\substack{d_1' \backslash q_i \\ d_2' \backslash q_j}} \mu(d_1')\mu(d_2') \left(\frac{1}{12} \cdot \frac{\left(\frac{q_i}{d_1'}, \frac{q_j}{d_2'}\right)^2}{\frac{q_i}{d_1'} \cdot \frac{q_j}{d_2'}} + \frac{1}{4} \right) \tag{11}$$

But if $q_i > 1$ for all i, then we can omit $\frac{1}{4}$ in (11). For a simplicity let

$$d_{ij}^0 = \prod_{\substack{p \backslash d_{ij} \\ p \nmid q_{ij}}} p, \quad d_{ij}^1 = \prod_{\substack{p \backslash d_{ij} \\ p \backslash q_{ij}}} p.$$

Obviously,

$$d_{ij}^0 = \prod_{\substack{p \backslash q_i, p \backslash q_j \\ \mathrm{ind}_p(q_i) = \mathrm{ind}_p(q_j)}} p, \quad d_{ij}^1 = \prod_{\substack{p \backslash q_i, p \backslash q_j \\ \mathrm{ind}_p(q_i) \neq \mathrm{ind}_p(q_j)}} p.$$

In what folllows, without loss of generality, we may suppose that d_1', d_2' are square-free. From it

$$\mu(d_1')\mu(d_2') = \mu(d_{12}'),$$

where

$$d_{12}' = \frac{d_1'}{(d_1', d_2')} \cdot \frac{d_2'}{(d_1', d_2')}. \tag{12}$$

Computing g.c.d. $\left(\frac{q_i}{d_1'}, \frac{q_j}{d_2'}\right)$ we must diminish $\operatorname{ind}_p(d_{ij})$ not only for prime $p \backslash (d_1', d_2')$ but also for this p if, e.g. $p \backslash d_{ij}$, $p \backslash d_1'$, $p \nmid d_2'$ and $\operatorname{ind}_p(q_i) \leqq \operatorname{ind}_p(q_j)$. From it we have

$$\left(\frac{q_i}{d_1'}, \frac{q_j}{d_2'}\right) = \frac{d_{ij}}{(d_1', d_2')} \cdot \frac{1}{(d_{12}', d_{ij}^0)} \cdot \frac{1}{(d_{12}', d_{ij}^1)^*}. \tag{13}$$

Here $(d_{12}', d_{ij}^1)^* = \Pi p$, where p ranges over all prime divisors of (d_{12}', d_{ij}^1) for which:

if $\operatorname{ind}_p(q_i) < \operatorname{ind}_p(q_j)$, then $p \Big\backslash \frac{d_1'}{(d_1', d_2')}$ and if $\operatorname{ind}_p(q_j) < \operatorname{ind}_p(q_i)$, then $p \Big\backslash \frac{d_2'}{(d_1', d_2')}$.

Next for the summation (11) we shall find all pairs of positive integers d_1', d_2' for which:

$$d_1',\ d_2' \text{ are square-free, } d_1' \backslash q_i,\ d_2' \backslash q_j \text{ and } d_{12}' = \text{constant}. \tag{14}$$

All these pairs we can obtain combining all admissible g.c.d. (d_1', d_2') with all admissible relatively prime decompositions of d_{12}' in the form (12). From (14) it follows that

$$(d_1', d_2') \Big\backslash \frac{d_{ij}^0 \cdot d_{ij}^1}{(d_{12}', d_{ij}^0 d_{ij}^1)} = \frac{d_{ij}^0}{(d_{12}', d_{ij}^0)} \cdot \frac{d_{ij}^1}{(d_{12}', d_{ij}^1)}.$$

In the decomposition (12) those primes have a stable place $\left(\text{either in } \frac{d_1'}{(d_1', d_2')} \text{ or in } \frac{d_2'}{(d_1', d_2')}\right)$ which divide $\frac{d_{12}'}{(d_{12}', d_{ij})}$, while places of primes which divide $(d_{12}', d_{ij}) = (d_{12}', d_{ij}^0) \cdot (d_{12}', d_{ij}^1)$ are arbitrary. The place of primes p which divide (d_{12}', d_{ij}^1) has an influence on the value $(d_{12}', d_{ij}^1)^*$ only. From it

$$\sum_{\substack{d_1', d_2' \\ \text{satisfy (14)}}} \frac{1}{(d_{12}', d_{ij}^1)^2}$$

$$= 2^{2\left(\frac{d_{ij}^0}{(d_{12}', d_{ij}^0)} \cdot \frac{d_{ij}^1}{(d_{12}', d_{ij}^1)}\right)} \cdot 2^{v((d_{12}', d_{ij}^0))} \cdot \prod_{p \backslash (d_{12}', d_{ij}^1)} \left(1 + \frac{1}{p^2}\right). \tag{15}$$

Simultaneously,

$$v\left(\frac{d_{ij}^0}{(d_{12}', d_{ij}^0)} \cdot \frac{d_{ij}^1}{(d_{12}', d_{ij}^1)}\right) + v((d_{12}', d_{ij}^0)) = v(d_{ij}) - v((d_{12}', d_{ij}^1)).$$

Consequently by (13) and (15) we can do sums (11) in a form

$$\sum_{i,j=1}^{n} \sum_{d'_{12}\backslash q_i q_j} \mu(d'_{12}) \cdot \sum_{\substack{d'_1, d'_2 \\ \text{satisfy (14)}}} \frac{1}{12} \cdot \frac{\left(\frac{q_i}{d'_1}, \frac{q_j}{d'_2}\right)^2}{\frac{q_{ij}}{d'_1} \cdot \frac{q_j}{d'_2}}$$

$$= \frac{1}{12} \sum_{i,j=1} \sum_{d'_{12}\backslash q_i q_j} \mu(d'_{12}) \cdot \frac{d'_{12}}{q_{ij}} \cdot \frac{1}{(d'_{12}, d^0_{ij})^2} \cdot \sum_{\substack{d'_1, d'_2 \\ \text{satisfy (14)}}} \frac{1}{(d'_{12}, d^1_{ij})^{*2}}$$

$$= \frac{1}{12} \sum_{i,j=1}^{n} \frac{2^{v(d_{ij})}}{q_{ij}} \sum_{d'_{12}\backslash q_i q_j} \mu(d'_{12}) d'_{12} \frac{1}{(d'_{12}, d^0_{ij})^2} \frac{1}{2^{v(d'_{12}, d^1_{ij})}} \prod_{p\backslash (d_\cdot, d^1_{ij})} \left(1 + \frac{1}{p^2}\right). \tag{16}$$

All functions from (16) are multiplicative in d'_{12}, therefore

$$= \frac{1}{12} \sum_{i,j=1}^{n} \frac{2^{v(d_{ij})}}{q_{ij}} \prod_{p\backslash q_i q_j} \left(1 - p \frac{1}{(p, d^0_{ij})^2} \frac{1}{2^{v((p, d^1_{ij}))}} \left(1 + \frac{1}{p^2}\right)^{v((p, d^1_{ij}))}\right).$$

Finally, examining three possible cases for $p\backslash q_i q_j$ we have shown (9). With regard to (10), we only state that

$$\sum_{i=1}^{n} \sum_{\substack{0 < x < q_i \\ (x, q_i) = 1}} \left(\frac{x}{q_i} - \frac{1}{2}\right) = 0$$

and also see (5), (6).

Thus the proof of Theorem 1 is finished.

To an attestation of Theorem 1, let p be a prime. For the sequence $p, p^2, \dots, p^n$ (8), (9), (10) are equal to

$$\frac{1}{6}\left(1 - \frac{1}{p^n}\right).$$

Remark 1. The particular case $n = 1$ of (9) first appears explicitly in [8] and implicitly in [9]. An other expression of (2) can be found in [7].

II.

In [1] Franel has shown that if we put $q_i = i$ for all i, i.e. if $\left\{\frac{x(j)}{q_{i(j)}}\right\}_{j-1}^{N}$, $N = \sum_{i=1}^{n} \varphi(q_i)$ is the sequence of Farey's fractions ordered by increasing magnitude, then the Riemann hypothesis is equivalent to

$$N\sum_{j=1}^{N}\left(\frac{2j-1}{2N}-\frac{x(j)}{q_{i(j)}}\right)^2 = O(N^{\frac{1}{2}+\varepsilon}).$$

In the following theorem we shall prove that if for a sequence $\{q_i\}_{i=1}^{\infty}$ of positive integers this sum is bounded, then Duffin—Schaeffer conjecture for $\{q_i\}_{i=1}^{\infty}$ is valid. This is a little step in the investigation into the coherence between Riemann hypothesis and Duffin—Schaeffer conjecture, which was conjectured by V. G. Sprindžuk in [4, p. 61].

Theorem 2. Let $\{q_i\}_{i=1}^{\infty}$ be a one-to-one sequence of positive integers and for any n let $N = \sum_{i=1}^{n} \varphi(q_i)$ and $\left\{\frac{x(j)}{q_{i(j)}}\right\}_{j=1}^{N}$ be the sequence of all reduced fractions from the interval [0, 1] whose denominators are from $\{q_i\}_{i=1}^{n}$ and which are ordered by increasing magnitude. If

$$N\sum_{j=1}^{N}\left(\frac{2j-1}{2N}-\frac{x(j)}{q_{i(j)}}\right)^2 = O(1),$$

then the sequence $\{q_i\}_{i=1}^{\infty}$ satisfies Duffin—Schaeffer conjecture with every nonnegative real function f for which the sequence $\{f(q_i)\}_{i=1}^{\infty}$ is nonicreasing. I.e. if

$$\sum_{i=1}^{\infty} \varphi(q_i) f(q_i) = +\infty,$$

then for almost all t and infinitely many i the diophantine inequality

$$\left|t-\frac{x}{q_i}\right| < f(q_i)$$

has an integral solution x relatively prime with q_i.

The proof is based on our theory of the so called "quick" sequences given in the paper [5]. By Definition 3 and 2 from [5]:

A sequence $\{x_i\}_{i=1}^{\infty}$ from the interval [0, 1] is said to be "quick" in [0, 1], if for every sequence $\{I_i\}_{i=1}^{\infty}$ of pairwise disjoint subintervals of [0, 1] for which $\{x_i\}_{i=1}^{\infty} \subset \bigcup_{i=1}^{\infty} I_i$ and the measure $\left|\bigcup_{i=1}^{\infty} I_i\right| = \sum_{i=1}^{\infty} |I_i| < |[0, 1]| = 1$ there exists a constant $c = c(\{I_i\}_{i=1}^{\infty})$ such that

$$\frac{M}{N} \geqq c > 0$$

for all N, where

$$M = M(N) = \operatorname{card}\{j;\ I_j \cap \{x_i\}_{i=1}^{N} \neq \emptyset,\quad j = 1, 2, \ldots\}.$$

A sequence $\{x_i\}_{i=1}^{\infty}$ from [0, 1] is said to be "eutaxic" (J. Lesca) if for every nonicreasing $\{z_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} z_i = +\infty$ we have for almost all $t \in [0, 1]$ that $|t - t_i| < z_i$ for infinitely many i is valid.

Corollary 1 from [5] states that every quick sequence is also eutaxic. This and the following lemma imply immediately Theorem 2.

Lemma. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points from [0, 1] and let

$$T_N = \frac{1}{N}\left(\int_0^1 R_N^2(t)\,\mathrm{d}t\right)^{\frac{1}{2}} = \left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{2i-1}{2N} - y_i\right)^2 + \frac{1}{12N^2}\right)^{\frac{1}{2}}$$

denote L^2 discrepancy, where $y_1, y_2, \ldots, y_N$ are the numbers $x_1, x_2, \ldots, x_N$ ordered into a nondecreasing sequence.

If $T_{N_n} = O(N_n^{-1})$, then the sequence $\{x_i\}_{i=1}^{\infty}$ is $\{N_n\}$ — quick, cf. [5, Definition 4].

Proof. For simplicity, let us assume $y_i = x_i$ for $i = 1, 2, \ldots, N$. Let us take an arbitrary system $\{I_i\}_{i=1}^{\infty}$ of pairwise disjoint subintervals of [0, 1] such that $\{x_i\}_{i=1}^{\infty} \subset \bigcup_{i=1}^{\infty} I_i$ and $\sum_{i=1}^{\infty} |I_i| < 1$. Let us denote

$$I_j \cap \{x_i\}_{i=1}^{N} = \{x_{l_j} \leqq x_{l_j+1} \leqq \ldots \leqq x_{r_j}\}$$

$$L = \operatorname{card}\{j;\ l_j \neq r_j,\quad j = 1, 2, \ldots\}$$

and let us assume that the initial segment $\{I_j\}_{j=1}^{M}$ of $\{I_i\}_{i=1}^{\infty}$ contains all intervals which have a nonzero intersection with $\{x_i\}_{i=1}^{N}$. Clearly,

$$M = N - \sum_{j=1}^{M}(r_j - l_j) = N - N\sum_{j=1}^{M}\left(\frac{2r_j - 1}{2N} - \frac{2l_j - 1}{2N}\right)$$

$$= N - N\sum_{j=1}^{M}(x_{r_j} - x_{l_j}) - N\sum_{j=1}^{M}\left(\frac{2r_j - 1}{2N} - x_{r_j}\right) - \left(\frac{2l_j - 1}{2N} - x_{l_j}\right).$$

Applying Cauchy inequality,

$$\frac{M}{N} \geqq 1 - \sum_{j=1}^{M}(x_{r_j} - x_{l_j}) - (2L)^{\frac{1}{2}}\left(\sum_{j=1}^{N}\left(\frac{2j-1}{2N} - x_j\right)^2\right)^{\frac{1}{2}}$$

$$\geqq 1 - \sum_{i=1}^{\infty}|I_i| - \left(\frac{2L}{N}\right)^{\frac{1}{2}} \cdot \left(N\sum_{j=1}^{N}\left(\frac{2j-1}{2N} - x_j\right)^2\right)^{\frac{1}{2}} \geqq 1 - \sum_{i=1}^{\infty}|I_i| - \left(\frac{2L}{N}\right)^{\frac{1}{2}} \cdot N \cdot T_N \tag{17}$$

If we replace L by M in (17) we can see a quadratic inequality for $\left(\frac{M}{N}\right)^{\frac{1}{2}}$, from it

$$\left(\frac{M}{N}\right)^{\frac{1}{2}} \geqq \frac{\sqrt{2}}{2}\left(-NT_N + \left((NT_N)^2 + 2\left(1 - \sum_{i=1}^{\infty} |I_i|\right)\right)^{\frac{1}{2}}\right).$$

Since by assumptions $NT_N \leqq c < +\infty$, $1 - \sum_{i=1}^{\infty} |I_i| > 0$, this proves Lemma.

Finally we notice

Remark 2. It holds

$$\frac{1}{4\pi^2}\left(\sum_{i=1}^{n} \mu(q_i)\right)^2 \leqq N \sum_{j=1}^{N}\left(\frac{2j-1}{2N} - \frac{x(j)}{q_{i(j)}}\right)^2 + \frac{1}{12} \leqq \left(\sum_{i=1}^{n} 2^{v(q_i)}\right)^2. \tag{18}$$

The left inequality follows from (according to E. Landau)

$$\left|\sum_{i=1}^{n} \mu(q_j)\right| = \left|\sum_{i=1}^{n} \sum_{\substack{0<x<q_j \\ (x,q_j)=1}} e^{2\pi i \cdot \frac{x}{q_j}}\right| = \left|\sum_{j=1}^{N} e^{2\pi i \cdot \frac{x(j)}{q_{i(j)}}}\right|$$

$$= \left|\sum_{j=1}^{N} e^{-\pi i \cdot \frac{2j-1}{2N}} \cdot \left(e^{2\pi i\left(\frac{2j-1}{2N} - \frac{x(j)}{q_{i(j)}}\right)} - 1\right)\right|$$

$$\leqq \sum_{j=1}^{N} \left|e^{2\pi i\left(\frac{2j-1}{2N} - \frac{x(j)}{q_{i(j)}}\right)} - 1\right| = 2\sum_{j=1}^{N}\left|\sin \pi\left(\frac{2j-1}{2N} - \frac{x(j)}{q_{i(j)}}\right)\right|$$

$$\leqq 2\pi \sum_{j=1}^{N}\left|\frac{2j-1}{2N} - \frac{x(j)}{q_{i(j)}}\right| \leqq 2\pi (N)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{N}\left(\frac{2j-1}{2N} - \frac{x(j)}{q_{i(j)}}\right)^2\right)^{\frac{1}{2}}.$$

The right inequality follows from (11) and from

$$\left|\sum_{i=1}^{n} \sum_{d \backslash q_i} \mu(d)\left\{\frac{q_i t}{d}\right\}\right| \leqq \sum_{i=1}^{n} \sum_{d \backslash q_i} |\mu(d) = \sum_{i=1}^{n} 2^{v(q_i)}.$$

Applying (18) we see that the expression $O(1)$ in Theorem 2 is not necessary since we can construct a sequence $\{q_i\}_{i=1}^{\infty}$ such tha $\frac{\varphi(q_i)}{q_i} \geqq c > 0$ for all i, from it $\{q_i\}_{i=1}^{\infty}$ holds Duffin—Schaeffer conjecture automatically, and such that $\left|\sum_{i=1}^{n} \mu(q_i)\right| \to +\infty$ as $N \to \infty$. Thus in the next investigation it is necessary to replace $O(1)$ by $O(\psi(N))$ for some $\psi(N) \to +\infty$ as $N \to \infty$.

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SÚHRN

POUŽITIE FRANELOVHO INTEGRÁLU

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V práci je okrem iného kokázané, že ak usporiadame racionálne čísla x/q_i, $0 < x < q_i$, $(x, q_i) = 1$, $i = 1, 2, \ldots, n$ do neklesajúcej postupnosti $x(j)/q_{i(j)}$, $j = 1, 2, \ldots, \sum_{i=1}^{\infty} \varphi(q_i) = N$ a ak

$$\sum_{j=1}^{N} \left(\frac{2j-1}{2N} - \frac{x(j)}{q_{i(j)}}\right)^2 = O(N^{-1})$$

potom postupnosť $\{q_i\}_{i=1}^{\infty}$ spĺňa Duffinovu—Schaefferovu hypotézu s každou nerastúcou postupnosťou $\{f(q_i)\}_{i=1}^{\infty}$ reálnych čísel.

РЕЗЮМЕ

ПРИМЕНЕНИЕ ИНТЕГРАЛА ФРАНЕЛА

Ото Штраух, Братислава

В работе между прочим показано, что если $\{q_i\}_{i=1}^{\infty}$ последовательность натуральных чисел удовлетворяют условию

$$\sum_{h=1}^{\infty} \frac{1}{h^2} \left| \sum_{i=1}^{n} \frac{\varphi(q_i)}{\varphi\left(\frac{q_i}{(h, q_i)}\right)} \mu\left(\frac{q_i}{(h, q_i)}\right) \right|^2 \leqq c < +\infty$$

(φ — функция Эйлера, μ — функция Мёбиуса), последовательность положительных действительных чисел $\{f(q_i)\}_{i=1}^{\infty}$ невозрастающая и

$$\sum_{i=1}^{\infty} \varphi(q_i) f(q_i) = +\infty ,$$

то для почти всех t диофантого неравенство

$$\left| t - \frac{x}{q_i} \right| < f(q_i)$$

имеет целочисленное решение x для бесконечно многих i, такое, что x, q_i — взаимно простые.

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FOURIER TRANSFORMS AND THE P.N.T. EROR TERM (ABSTRACT)

JIŘÍ ČÍŽEK, Plzeň

For every function $\omega(x)$ such that both $\omega(x)$ and $\frac{1}{6}\log x - \omega(x)$ are positive and increasing for $x \in \langle 3; +\infty)$, the prime number theorem with the error term, i.e. the assertion

$$\pi(x) = \int_2^x \frac{du}{\log u} + O\{x \exp(-\omega(x))\}, \; x \geq 3 \tag{1}$$

is implied by the assertion

$$g(x) = \sum_{n \leq x} \Lambda(n) \log \frac{x}{n} = x + O(\{x \exp(-2\omega(x))\}, \; x \geq 3$$

where $\pi(x)$ is the number of primes less than or equal to x, $\Lambda(n) = \log p$ for $n = p^r$, where p is a prime, $r \in N$, $\Lambda(n) = 0$ otherwise (Λ is called von Mangoldt's function).

For $x > 0$ we have

$$g(x) = -\frac{1}{2\pi i} \int_{2-\infty i}^{2+\infty i} \frac{x^s}{s^2} \cdot \frac{\zeta'(s)}{\zeta(s)} \mathrm{d}s = I_1(x) + I_2(x),$$

where

$$I_1(x) = \frac{1}{2\pi i} \int_{2-\infty i}^{2+\infty i} \frac{x^s \, \mathrm{d}s}{s^2(s-1)}, \quad I_2(x) = -\frac{1}{2\pi i} \int_{2-\infty i}^{2+\infty i} \frac{x^s}{s^2} h(s) \, \mathrm{d}s,$$

and $h(s) = \zeta'(s)/\zeta(s) + 1/(s-1)$ is an analytic function in the halfplane $\mathrm{Re}\, s \geq 1$ (cf. [3]). By the technique of residues we can prove that $I_1(x) = x - \log x - 1$, $x \geq 1$. From Cauchy's theorem it follows that

$$I_2(x) = -\frac{1}{2\pi i} \int_{1-\infty i}^{1+\infty i} \frac{x^s}{s^2} h(s) \, \mathrm{d}s = -\frac{x}{2\pi} I_3(\log x),$$