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A CRITERION FOR UNIFORM DISTRIBUTION OF SEQUENCES AND A CLASS OF RIEMANN INTEGRABLE FUNCTIONS (PRELIMINARY ANNOUNCEMENT)

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In the paper J. Horbowicz: *Criteria for uniform distribution*, Indag. Math. 43 (1981), 301—307 the following criterion for uniform distribution of sequences of real numbers is proved:

Let f be a complex Riemann integrable function on [0, 1]. Let

$$\mu(Z(f)) = 0 \tag{1}$$

where $Z(f) = \{x \in [0, 1] : f(x) = 0\} = f^{-1}(0)$ (zero set of f) and μ is the Lebesgue measure. Then the sequence $\{x_n\}_{n=1}^{\infty}$ of numbers from $[0, 1)$ is uniformly distributed if and only if for each interval $[a, b) \subset [0, 1)$ we have

$$\lim_{\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n)\, \chi_{[a,b)}(x_n) = \int_a^b f(t)\, \mathrm{d}t$$

($\chi_{[a,b)}$ stands for the characteristic function of $[a, b)$).

This criterion represents a generalization of the following well-known fact from the theory of uniform distribution (cf. L. Kuipers—H. Niederreiter: *Uniform Distribution of Sequences*, 1974 p. 7 Exercise 1.14): Denote by $S([a, b); N)$ the sum $\sum_{n \leq N,\, x_n \in [a,b)} x_n$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ is uniformly distributed if and only if we have (for each $[a, b) \subset [0, 1)$)

$$\lim_{N\to\infty} \frac{S([a, b); N)}{N} = \frac{b^2 - a^2}{2}.$$

This result follows from the Horbowicz criterion as a special case. Indeed, putting $f(t) = t$, we get

$$\frac{1}{N} \sum_{n=1}^{N} f(x_n)\, \chi_{[a,b)}(x_n) = \frac{S([a, b); N)}{N}$$

and

$$\int_a^b f(t)\,dt = \int_a^b t\,dt = \frac{b^2 - a^2}{2}.$$

Š. Porubský posed the question whether the condition (1) can be replaced (in the criterion of Horbowicz) by the condition

$$(1') \qquad m(Z(f)) = 0,$$

where m denotes the Jordan measure. The answer to this question is in the positive. It can be namely shown that for an arbitrary Riemann integrable function f we have

$$(2) \qquad \mu(\overline{Z(f)}) = \mu(Z(f)$$

($\bar{M}$ denotes the closure of M in [0, 1]). From (2) the positive answer to the mentioned question follows at once.

Further the question arises how large is the class $H(0, 1)$ of all functions $f \in R(0, 1)$ ($R(0, 1)$ denotes the set of all Riemann integrable functions on [0, 1]) satisfying the condition (1). We answer this question from the topological point of view. In what follows $R(0, 1)$ denotes the linear normed space (with the sup-norm) of all real Riemann integrable functions on [0, 1]. This is evidently a Banach space since the convergence in this space coincides with the uniform convergence which preserves the Riemann integrability. (We restrict ourselves only to real functions but the following result can be easily extended also for complex functions).

Theorem. The set $R(0, 1) \setminus H(0, 1)$ is a nowhere dense set in $R(0, 1)$.

Hence $H(0, 1)$ is a residual set in the space $R(0, 1)$ and therefore the functions that can be used in the criterion of Horbowicz are typical in $R(0, 1)$ (in the sense of the contemporary theory of real functions — see A. M. Bruckner *Differentiation of Functions*, Springer—Verlag, Berlin—Heidelberg—New York, 1978,. (p. 50).

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SÚHRN

O ISTOM KRITÉRIU ROVNOMERNÉHO ROZDELENIA POSTUPNOSTÍ A TRIEDE RIEMANNOVSKY INTEGROVATEĽNÝCH FUNKCIÍ

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Nech $R(0, 1)$ označuje priestor všetkých funkcii integrovateľných v Riemannovom zmysle na $\langle 0, 1\rangle$ s metrikou

$$d(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|$$

Možno dokázať, že množina všetkých funkcií $f \in R(0, 1)$, ktoré sa dajú použiť v kritériu z prace J. Horbowicza, má tvar $R(0, 1) \setminus M$, kde M je množina riedka v $R(0, 1)$.

РЕЗЮМЕ

ОДИН ПРИЗНАК ДЛЯ РАВНОМЕРНОГО РАССПРЕДЕЛЕНИЯ ПОСЛЕДОВАТЕЛЬНОСТЕЙ И ОПРЕДЕЛЕННЫЙ КЛАСС ФУНКЦИЙ ИНТЕГРИРУЕМЫХ В СМЫСЛЕ РИМАНА

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Пусть $R(0, 1)$ обозначает пространство всех функций интегрируемых в смысле Римана на интервале $\langle 0, 1\rangle$ с метрикой $d(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|$. Можно показать, что множество всех $f \in R(0, 1)$, которые можно использовать в признаку работы Горбовича, имеет форму $R(0, 1) M$, где M нигде не плотное множество в $R(0, 1)$.