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METHOD FOR NUMERICAL SOLVING THE CAUCHY'S PROBLEM IN ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction

Let us consider the Cauchy's problem in the following formm:

$$\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \end{aligned} \tag{1}$$

and suppose that:

1. $y_0 \neq 0$ and the solution $y(x) \neq 0$ for all considering x.
2. The Cauchy's problem (1) has the unique solution $y(x)$, passing through the point $[x_0, y_0]$.
3. The solution $y(x)$ can be "better approximate" by the exponential function of the form:

$$F(x) = e^{P_m(x)}, \tag{2}$$

where $P_m(x)$ is a polynom of degree m, i.e.:

$$P_m(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0 . \tag{3}$$

The expression "better approximate" means that for all x from the interval of the solution of the Cauchy's, problem and for polynom $Q_m(x)$ it holds

$$\max_{\in\langle\ \ \rangle\ x\ a,b} |y(x) - e^{P_m(x)}| < \max_{x \in \langle a, b\rangle} |y(x) - Q_0(x)| . \tag{4}$$

It means that we suppose that the solution of the problem (1) has an exponential character and it can be approximate by the exponential function in the form (2).

Thus we can write:

$$y(x) \approx F(x) \quad \text{or} \quad y(x) = F(x) + E$$

Futher we consider:

$$y(x) = F(x) \tag{5}$$

$$\frac{d}{dx} y(x) = f(x, y) = \frac{d}{dx} F(x) = F'(x) \tag{6}$$

Also we can consider the higher derivatives of $f(x)$ as follows:

$$f^{(n)} = F^{(n+1)} \qquad \text{for } n = 0, 1, \ldots, m - 1$$

Those derivatives are the total derivatives of the function $f(x, y)$ and for them it is true that:

$$\frac{d}{dx} f = \frac{\partial f}{\partial x} + f \cdot \frac{\partial f}{\partial y} = f^{(1)} \tag{8}$$

and

$$f^{(n+1)} = \frac{\partial f^{(n)}}{\partial x} + f \cdot \frac{\partial f^{(n)}}{\partial y} \tag{9}$$

As usual consider the interval $\langle a, b \rangle$ of the Cauchy's problem solution. The interval $\langle a, b \rangle$ is divided into n equidistant parts, i.e.:

$$\begin{aligned} x_0 &= a \\ x_i - x_{i-1} &= h \qquad \text{for } i = 1, 2, \ldots, n \\ x_n &= b \end{aligned} \tag{10}$$

The value h is the stepsize of discretization of the solution. Our task is to approximate the solution $y(x)$ in the points x_i for $i = 0, 1, \ldots, n$ and for the results we use the symbol y_i. Thus the problem was reduced to determine the unknown coefficients of the polynom $P_m(x)$.

2. The numerical method

Deriving the function $F(x)$ m-times (for simplicity the subscript m and the argument x in the polynom $P_m(x)$ is omitted), we obtain:

$$\begin{aligned} F &= e^P \\ F' &= e^P \cdot P \\ F'' &= e^P \cdot (P'' + (P)^2) \\ F''' &= e^P \cdot (P''' + 3P''P' + (P')^3) \end{aligned} \tag{11}$$

and the high order derivatives (m) of $F(x)$, using the Faa di Bruna's formula, can be written:

$$F^{(m)} = \left[\sum \frac{m!}{k_1!\, k_2 \dots k_m!} \left(\frac{P'}{1!}\right)^{k_1} \left(\frac{P''}{2!}\right)^{k_2} \dots \left(\frac{P^{(m)}}{m!}\right)^{k_m} \right] \cdot e^P, \tag{12}$$

where the summation goes throught all numbers $k_1, k_2, \dots, k_m$ which satisfy the following conditions:

$$k_1 + k_2 + \dots + k_m = k \quad \text{and} \quad k_1 + 2k_2 + 3k_3 + \dots + mk_m = m, \tag{13}$$

where $k = 1, 2, \dots, m$ and considering that $F(x) \neq 0$, then:

$$\frac{F^{(n)}}{F} = \sum \frac{n!}{k_1!k_2 \dots k_n!} \left(\frac{P'}{1!}\right)^{k_1} \left(\frac{P''}{2!}\right)^{k_2} \dots \left(\frac{P^{(n)}}{n!}\right)^{k_n} \tag{14}$$

The formula (14) is a polynomial function of n variables P', P'' ... $P^{(n)}$ and is in general non-linear. The only linear term in (14) is the term with the highest derivative and his coefficient is always 1. This fact is considered to be very important and all m derivatives of $P_m(x)$ can be found assuming that the total derivatives of function $f(x, y)$ are known. If it is assumed that the solution y_i of the Cauchy'ss problem exists (taking into account the initial value or using another numerical method) than by and discretizating of the system (14) in the point $x = x_i$, where $x_i = a + ih$, for all $n = 1, 2, 3, \dots, m$ we attain the system of nonlinear equations of n variables P'', P' ... $P^{(m)}$. Considering the system (14) for $n = 1$ we obtain the value of derivative P', for $n = 2$ the value of second derivative P'', etc. until $n = m$ and all derivatives $P_m^{(k)}(x_i)$ for $k = 1, 2, \dots m$ are found.

Further step is to find the unknown coefficients of the polynom $P_m(x)$ by using all m derivatives of $P_m(x)$. Thus.

$$\begin{aligned}
P'_m(x) &= ma_m x^{m-1} + (m-1)a_{m-1}x^{m-2} + \dots + a_1 \\
P''_m(x) &= m(m-1)a_m x^{m-2} + (m-1)(m-2)a_{m-1}x^{m-3} + \dots + 2a_2 \\
&\dots \\
P_m^{(n)}(x) &= m_{[m-n]}a_m x^{m-n} + (m-1)_{[m-n]}a_{m-1}x^{m-n-1} + \dots + n!a_{m-n} \\
&\dots \\
P_m^{(m)}(x) &= m!a_m,
\end{aligned} \tag{15}$$

where $m_{[k]}$ is decreasing factorial $m_{[k]} = m(m-1) \dots (m-k+1)$.

Discretizating and considering the system (15) in the point $x = x_i$, the linear equation system with unknowns $a_m, a_{m-1}, \dots, a_1$ is obtained. The system (15) is in the upper diagonal form and can be solved explicitly by substitution method.

By this procedure all unknown parameters of the polynom $P_m(x)$ were obtained except the parameter a_0. This parameter can be found as follows:

$$y(x) = \exp(a_0 + a_1 + \dots + a_m x^m) = q \cdot \exp(a_1 x + \dots + a_m x^m) \tag{16}$$

and discretizating in the point $x = x_i$ we can obtain

$$q = \frac{y_i}{\exp(a_1 x_i + \ldots + a_m x_i^m)} \tag{17}$$

and by (17) the last parameter of the method was found. The next value of the solution, i.e. y_{i+1} can be obtained as

$$y_{i+1} = q \cdot \exp(a_1(x_i + h) + a_2(x_i + h)^2 + \ldots + a_m(x_i + h)^m). \tag{18}$$

Formula (18) is the explicit formula for solving the Cauchy's problem. It has some advantages, namely for steep functions. The disadvantage of these methods is that the total derivatives of function $f(x, y)$ have to be exactly determined. But disregarding the disadvantages with high derivatives, the above described method is a new way of finding the numerical solution of a special type of Cauchy's problem.

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SÚHRN

NUMERICKÁ METÓDA RIEŠENIA CAUCHYHO PROBLÉMU V OBYČAJNÝCH DIFERENCIÁLNYCH ROVNICIACH

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V práci sa uvádza postup odvodenia numerickej metódy riešenia Cauchyho problému 1. rádu v prípade, že riešenie možno aproximovať exponenciálnou funkciou všeobecného tvaru:

$$y(x) = e^{P_m(x)}$$

kde $P_m(x)$ je polynom stupňa m. V práci sú opísané jednokrokové explicitné metódy, ktoré využívajú deriváciu pravej strany diferenciálnej rovnice.

РЕЗЮМЕ

ЧИСЛЕННЫЙ МЕТОД РЕШЕНИЯ ЗАДАЧИ КОШИ В ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛНЫХ УРАНЕНИЯХ

Йосеф Данчо, Братислава

В статье изучается одной численный метод получения решения задачи Коши первого порядка в случае, когда решение имеет экспоненциальный характер следующего вида:

$$y(x) = e^{P_m(x)},$$

где $P_m(x)$ полином степени m. В статье опсаны одношаговые явные методы, использующие производную правой стороны дифференциального уравнения.