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ON POINTS OF ABSOLUTE CONTINUITY OF DISCONTINUOUS FUNCTIONS

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A point $p \in (a, b)$, $-\infty \leq a < b \leq \infty$, is said to be a point of absolute continuity of a function f: $(a, b) \to R$ (R — the real line) if there is such a $\delta > 0$ that the function f is absolutely continuous on the interval $I = [p - \delta, p + \delta] \subset \subset (a, b)$.

Let $G(f)$ be the set of all points of absolute continuity of f. Then $G(f)$ is obviously open in (a, b) and $N(f) = (a, b) - G(f)$ is closed in (a, b).

In [6] a new method has been introduced. This method enables to prove the following statements.

Theorem HŠ. Let a continuous function f: $(a, b) \to R$ be symmetrically differentiable on the interval (a, b). Then $N(f)$ is nowhere dense in (a, b). (See [3].)

Theorem Š. Let a continuous function f: $(a, b) \to R$ be approximately differentiable on the interval (a, b). Then $N(f)$ is a nowhere dense set in (a, b). (See [6].)

The aim of the present paper is to show that the above statements hold also for some classes of discontinuous functions. We shall use the following notions: $D^s f(x)$ ($D_s f(x)$) denotes upper (lower) symmetric derivative of f at $x \in (a, b)$, i.e. $D^s f(x) = \limsup_{h \to 0} (f(x + h) - f(x - h))/(2h)$ $(D_s f(x) = \liminf_{h \to 0} (f(x + h) - f(x - h))/(2h))$. When $D^s f(x) = D_s f(x)$, whether finite or infinite, the common value is denoted by $f^s(x)$ and is called the symmetric derivative of f at x. The approximate derivative of f at x is denoted by $f'_{ap}(x)$.

According to [1] and [5] we introduce the next classes of functions:

$M_{-1} = \{f$: $(a, b) \to R$: f is measurable and $\liminf_{t \to x} f(t) \leq f(x) \leq \limsup_{t \to x} f(t)$ holds for each $x \in (a, b)\}$, and

$A = \{f$: $(a, b) \to R$: for each $x \in (a, b)$ there exists $f'_{ap}(x)$; if $\lim_{h \to 0+} f(x - h)$ exists, then it equals to $f(x)$ and if $\lim_{h \to 0+} f(x + h)$ exists, then it equals to $f(x)\}$.

We shall use the following modifications of the Mean Value Theorem.

Theorem E. Let $f \in M_{-1}$. If $a < \alpha < \beta < b$, then there exist points x_1 and x_2 in $[\alpha, \beta]$ such that $D_s f(x_1) \leq (f(\beta) - f(\alpha))/(\beta - \alpha) \leq D^s f(x_2)$. (See [1].)

Theorem P. Let $f \in A$. Then f'_{ap} has the Denjoy property, f'_{ap} is a Darboux function and f fulfils the Mean Value Theorem, i.e. $f(\beta) - f(\alpha) = f'_{ap}(\xi)(\beta - \alpha)$, where $a < \alpha < \xi < \beta < b$. (See [5].)

In the proof of Theorem 1 we shall use the following statement, which gives a sufficient condition for functions $D^s f$ and $D_s f$ to be in Baire class one.

Theorem F. Let f be approximately continuous and let $f^s(x)$ exist everywhere with the exception of points of a countable set. Then functions $D^s f$, $D_s f$ and f^s belong to the Baire class one. (See [2].)

Theorem 1. Let f: $(a, b) \to R$ be an approximately continuous function and let $D^s f$ and $D_s f$ be finite functions. Let $f^s(x)$ exist everywhere with the exception at most of a countable set. Then the set $N(f)$ is nowhere dense in (a, b).

Proof. Denote by $C(g)$ $(D(g))$ the set of all continuity (discontinuity) points of a function g: $(a, b) \to R$. Since $D_s f$ and $D^s f$ are functions in the first Baire class (Theorem F), sets $D(D_s f)$ and $D(D^s f)$ are sets of the first Baire category in (a, b). If $x \in C(D_s f) \cap C(D^s f)$, then there exists an interval $I = [x - \delta, x + \delta] \subset \subset (a, b)$ such that $D_s f$ and $D^s f$ are bounded on I. Hence there is a $K > 0$ such that $|D_s f(y)| \leq K$ and $|D^s f(y)| \leq K$ hold for each $y \in I$. Let $\alpha, \beta \in I$. Then Theorem E implies that there are points $x_1, x_2 \in [\alpha, \beta]$ such that $D_s f(x_1) \leq (f(\beta) - f(\alpha))/ /(\beta - \alpha) \leq D^s f(x_2)$, hence $|f(\beta) - f(\alpha)| \leq K|\beta - \alpha|$. Consequently f is a Lipschitz function on I, $C(D_s f) \cap C(D^s f) \subset G(f)$ and $N(f) \subset D(D_s f) \cup D(D^s f)$. Since $N(f)$ is a closed set of the first Baire category in (a, b), it is nowhere dense in (a, b).

It is known that the symmetric derivative of an arbitrary function belongs to the Baire class one (see [4]). This fact and Theorem E, using method of the proof of Theorem 1, imply the following statement.

Theorem 2. Let the function $f \in M_{-1}$ be symmetrically differentiable on the interval (a, b). Then the set $N(f)$ is nowhere dense in (a, b).

Analogously we can treat approximately differentiable functions. It is known that any approximate derivative belongs to the first Baire class (see [5]). This fact and Theorem P, using method of the proof of Theorem 1, imply the following assertion.

Theorem 3. Let the function $f \in A$ be approximately differentiable on the interval (a, b). Then the set $N(f)$ is nowhere dense in (a, b).

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SÚHRN

O BODOCH ABSOLÚTNEJ SPOJITOSTI NESPOJITÝCH FUNCKIÍ

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Predložená práca úzko nadväzuje na citované články [3] a [6]. Dokazuje sa, že niektoré výsledky týkajúce sa množín bodov absolútnej spojitosti spojitých funkcií je možné rozšíriť aj na isté triedy nespojitých funkcií.

РЕЗЮМЕ

О ТОЧКАХ АБСОЛЮТНОЙ НЕПРЕРЫВНОСТИ РАЗРЫВНЫХ ФУНКЦИЙ

Павел Костырко, Братислава

Работа узко примыкается к работам [3] и [6]. Показано, что некоторые результаты касающиеся множеств точек абсолютной непрерывности непрерывных функций расспространимы на некоторые классы разрывных функций.