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ADDITION TO „SOME QUESTIONS OF QUASICONTINUITY"

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In [1] the following theorem was introduced.

Theorem A. (Theorem 4 in [1]) Let X be a first countable Hausdorff topological space and let Y be a metric space. If a multifunction $F: X \to Y$ is quasicontinuous at a point x_0 and uniformly compact near x_0, if a set $F(x_0)$ is closed and if a function $f: X \times Y \to (-\infty, +\infty)$ is continuous at (x_0, y) for any $y \in F(x_0)$, a multifunction $M: X \to Y$ defined by the equality

$$M(x) = \{y \in F(x): f(x, y) = \sup\{f(x, y): z \in F(x)\}\}$$

is upper semi-quasicontinuous at x_0.

Recall that a multifunction $F: X \to Y$ is said to be uniformly compact near x_0 if there exists neighbourhood U of x_0 such that the set $\overline{\bigcup_{x \in U} F(x)}$ is compact.

The proof of this theorem was based on a characterization of the quasicontinuity which can be found in [2]. The majority of the assumptions given on spaces X, Y and on the multifunction F in the mentioned theorem are needed for the sake of this characterization.

The main aim of this paper is to give another proof of the introduced result which allows to omit several assumptions. Speaking more precisely, the following theorem is valid.

Theorem B. Let X, Y be arbitrary topological spaces, let a multifunction F be quasicontinuous at a point $x_0 \in X$, let $F(x_0)$ be compact and let a function f be continuous at (x_0, y) for any $y \in F(x_0)$. Then the multifunction M is upper semi-quasicontinuous at x_0.

The connection of this theorem with mathematical programming is evident since the value of the multifunction M can be interpreted as the set of optimal solutions when the objective function f and the set F of constraints are given. A preservation of a certain type of the generalized continuity as e.g. the quasicontinuity, can be understood as a certain stability of this solution set.

Before giving the proof, let us remark that a multifunction $F: X \to Y$ is said to be upper semi-quasicontinuous at x_0 if for any open set $V \subset F(x_0)$ and for any neighbourhood U of a point x_0 there exists a nonempty open set $G \subset U$ such that $F(x) \subset V$ for any $x \in G$. A multifunction F is said to be quasicontinuous at x_0 if for any open sets V_1, V_2 and U such that $F(x_0) \subset \subset V_1$, $F(x_0) \subset V_2 \neq \emptyset$ and $x_0 \in U$ there exists a nonempty open set $G \subset U$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for any $x \in G$.

Proof. Let U_0, V_0 be open sets such that $x_0 \in U_0$ and $M(x_0) \subset V_0$. Suppose that $F(x_0) \subset V_0$ does not hold because in this case the upper semi-quasicontinuity of M at x_0 is obvious.

If we denote V_0' the complement of the set V_0, then the sets $F(x_0)$ and $K = = F(x_0) \cap V_0'$ are compact and from the continuity of f we obtain the existence of $y_{MAX} \in M(x_0)$ and $y_{max} \in K$ such that

$$f(x_0, y_{MAX}) = \sup\{f(x_0, y) \colon y \in F(x_0)\} > \\ > \sup\{f(x_0), y) \colon y \in K\} = f(x_0, y_{max}).$$

Denote $\varepsilon = f(x_0, y_{MAX}) - f(x_0, y_{max}) > 0$.

Furthermore, from the continuity of f it follows that for any $\bar{y} \in K$ there exist neighbourhoods $U_{\bar{y}}$ of x_0 and $V_{\bar{y}}$ of $\bar{y}$ such that we have

$$f(x, y) < f(x_0, \bar{y}) + \frac{\varepsilon}{3} \tag{1}$$

for any $(x, y) \in U_{\bar{y}} \times V_{\bar{y}}$.

Since the set K is compact, there are $\bar{y}_1, \cdots, \bar{y}_n \in K$ such that $K \subset V_1 = = \bigcup_{i=1}^{n} V_{\bar{y}_i}$.

It is evident that the set $L = F(x_0) \cap V_1'$ is also compact and, moreover,

$$L = F(x_0) \cap V_1' \subset F(x_0) \cap K' = F(x_0) \cap (F'(x_0) \cup V_0) \subset V_0.$$

For any $\tilde{y} \in L$ there exist neighbourhoods $U_{\tilde{y}}$ of x_0 and $V_{\tilde{y}}$ of $\tilde{y}$ such that have

$$f(x_0, \tilde{y}) - \frac{\varepsilon}{3} < f(x, y) \tag{2}$$

for any $(x, y) \in U_{\tilde{y}} \times V_{\tilde{y}}$.

Neighbourhoods $V_{\tilde{y}}$ can be chosen in such a way that $V_{\tilde{y}} \subset V_0$. Also, since L is compact, we obtain $L \subset \bigcup_{j=1}^{m} V_{\tilde{y}_j}$. If we denote $V_2 = \bigcup_{j=1}^{m} V_{\tilde{y}_j}$, then $V_2 \subset V_0$.

Consider a neighbourhood $V_{y_{MAX}}$ of the point y_{MAX} considered as a point from the set L. Then we have

$$F(x_0) \cap V_{y_{MAX}} \neq \emptyset \quad \text{and} \quad F(x_0) \subset V_1 \cup V_2.$$

Denote

$$\tilde{U} = U_0 \cap U_{y_{\mathrm{MAX}}} \cap \left(\bigcap_{i=1}^{n} U_{\bar{y}_i}\right) \cap \left(\bigcap_{j=1}^{m} U_{\bar{y}_j}\right),$$

where $U_{\bar{y}_i}$, $U_{\bar{y}_j}$ and $U_{y_{\mathrm{MAX}}}$ are the neighbourhoods of x_0 corresponding to $V_{\bar{y}_i}$, $V_{\bar{y}_j}$ and $V_{y_{\mathrm{MAX}}}$ respectively.

From the quasicontinuity of F at x_0 there follows the existence of a nonempty open set $G \subset \tilde{U}$ such that $F(x) \subset V_1 \cup V_2$, $F(x) \cap V_{y_{\mathrm{MAX}}} \neq \emptyset$ for any $x \in G$. Therefore for any $x \in G$ there is $y_x \in F(x) \cap V_{y_{\mathrm{MAX}}}$ and according to (2) we have

$$f(x_0, y_{\mathrm{MAX}}) - \frac{\varepsilon}{3} < f(x, y_x). \tag{3}$$

We shall show that if $y \in F(x) \cap V_0'$, then y must belong to V_1. The inclusion $V_0' \subset V_2'$ is valid for the sake of the choice of $V_{\bar{y}_j}$. From this inclusion we see that such y cannot belong to V_2 and therefore $y \in V_1$ is valid. Thus $y \in V_{\bar{y}_i}$ for some i and according to (1) we have

$$f(x, y) < f(x_0, \bar{y}_i) + \frac{\varepsilon}{3} \leq f(x_0, y_{\max}) + \frac{\varepsilon}{3}. \tag{4}$$

Combining (3) and (4) we obtain

$$f(x, y) < f(x_0, y_{\max}) + \frac{\varepsilon}{3} < f(x_0, y_{\mathrm{MAX}}) - \frac{\varepsilon}{3} < f(x, y_x).$$

Thus we see that y cannot belong to $M(x)$ and therefore $M(x) \subset V_0$. The upper semi-quasicontinuity of M at x_0 is proved.

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SÚHRN

DODATOK K „NIEKTORÉ OTÁZKY KVÁZISPOJITOSTI"

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V práci sa uvádza nový dôkaz vety 4 z [1], ktorá pojednáva o množine optimálnych riešení popísanej pomocou určitej multifunkcie a o kvázispojitosti tejto multifunkcie. Tento dôkaz dovoľuje vypustiť niektoré predpoklady uvedené v [1] a rozširuje platnosť vety na ľubovoľné topologické priestory.

РЕЗЮМЕ

ДОБАВЛЕНИЕ К «НЕКОТОРЫЕ ВОПРОСЫ КВАЗИ-НЕПРЕРЫВНОСТИ»

ОНДРЕЙ НАТЭР, Братислава

В работе даётся новое доказательство теоремы 4 из [1], которая трактует множество оптимальных решений описанное при помощи определённой мультифункции, и рассматривать квази-непрерывность этой мультифункции. Это доказательство позволяет пропустить некоторые условия, приведенные в [1], и расширяет действие теоремы на произвольные топологические пространства.

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ON CERTAIN CHARACTERIZATION OF GENERALIZED CONTINUITY OF MULTIFUNCTIONS

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In the first part of the present paper a concept of the so called $\mathcal{S}$-continuity for multifunctions is introduced. This concept, which was introduced for functions in [2], includes several generalizations of the continuity. The aim of the paper is to give a characterization of the $\mathcal{S}$-continuity of a multifunction with the help of the $\mathcal{S}$-continuity of functions from a certain family corresponding to this multifunction.

Several theorems are proved using the above-mentioned characterization. The majority of these results were already proved before, some of them in the weaker and some in the stronger form. But the news of the work is the approach which enables to exploit well-known properties of real functions and makes the proofs very simple.

If not specified, X, Y denote general topological spaces, R denotes, the set of real numbers with the usual topology and R^+, R_0^+ denote the set of all positive and nonnegative real numbers respectively.

1. $\mathcal{S}$-semicontinuity

In [2] the following concepts are introduced.

Definition 1. A family $\mathcal{S}_x$ of subsets of X is called a local sieve at a point $x \in X$ if:

1. $x \in A$ for any $A \in \mathcal{S}_x$,
2. $A \subset B$ and $A \in \mathcal{S}_x$ implies $B \in \mathcal{S}_x$,
3. $\mathcal{U}_x \subset \mathcal{S}_x$, where $\mathcal{U}_x$ denotes the system of all neighbourhoods of a point x.

Definition 2. A local sieve $\mathcal{S}_x$ is called strongly local if $A \cap U \in \mathcal{S}_x$ for any $A \in \mathcal{S}_x$ and for any $U \in \mathcal{U}_x$.

In everything that follows we shall consider only strongly local sieves.

Examples of the sieves which are not strongly local can be found in [2], where also the following concept is introduced.

Definition 3. If $\mathcal{S}_x$ is a local sieve at a point $x \in X$, we say the function f from X to Y is $\mathcal{S}$-continuous at x if $f^{-1}(V) \in \mathcal{S}_x$ for any neighbourhood V of the point $f(x)$.

If we consider real-valued functions we can introduce the concept of $\mathcal{S}$-semicontinuity which we shall call order $\mathcal{S}$-semicontinuity to distinguish this one from the $\mathcal{S}$- semicontinuity of multifunctions. In the following definitions we suppose that a local sieve $\mathcal{S}_x$ at a point $x \in X$ is given.

Definition 4. A function f: $X \to R$ is said to be order upper (lower) $\mathcal{S}$-semicontinuous at a point x if for any $r \in R^+$ there exists a set $A \in \mathcal{S}_x$ such that $f(z) < f(x) + r$ ($f(z) > f(x) - r$) for any $z \in A$.

Definition 5. A multifunction F: $X \to Y$ is said to be upper (lower) $\mathcal{S}$-semicontinuous at a point x if for any open set V such that $F(x) \subset V$ ($F(x) \cap V \neq \emptyset$) there exists a set $A \in \mathcal{S}_x$ such that $F(z) \subset V$ ($F(z) \cap V \neq \emptyset$) for any $z \in A$.

We denote the order upper $\mathcal{S}$-semicontinuity, the order lower $\mathcal{S}$-semicontinuity, the upper $\mathcal{S}$-semicontinuity, the lower $\mathcal{S}$-semicontinuity by o.u.$\mathcal{S}$-s.c., o.l.$\mathcal{S}$-s.c., u.$\mathcal{S}$-s.c., l.$\mathcal{S}$-s.c. repectively.

The corresponding notions of o.u.$\mathcal{S}$-s.c., o.l.$\mathcal{S}$-s.c., u.$\mathcal{S}$-s.c. and l.$\mathcal{S}$-s.c. on X are understood as o.u.$\mathcal{S}$-s.c., o.l.$\mathcal{S}$-s.c., u.$\mathcal{S}$-s.c. and l.$\mathcal{S}$-s.c. at any $x \in X$ respectively.

By means of a special selection of local sieves $\mathcal{S}_x$ we can obtain several known types of generalized continuity.

If $\mathcal{S}_x = \mathcal{U}_x$, we obtain the continuity with respect to the topology given on X. In this case the notation o.u.s.c, o.l.s.c., u.s.c. and l.s.c. will be used.

If $\mathcal{S}_x = \{A : x \in A, x \in \bar{A}^{\circ}\}$, we obtain the quasicontinuity. Here the symbols A° and $\bar{A}$ are used for the interior and for the closure of the set A respectively. The notations o.u.q.c., o.l.q.c., u.q.c. and l.q.c. will be used.

If $\mathcal{S}_x = \{A : x \in A, x \in (\bar{A})^{\circ}\}$, we obtain the almost continuity.

If $X = R^n$, then the approximate continuity can be obtained as the $\mathcal{S}$-continuity, where the local sieve $\mathcal{S}_x$ at a point x is formed by all the sets which contain x as a density point.

For the definitions of the above-mentioned concepts see [2], where all these sieves are proved to be strongly local, too.

2. Characterization of $\mathcal{S}$-semicontinuity

From this moment suppose $(Y, \mathcal{V})$ to be a uniform space and consider only multifunctions with nonempty values in Y.

Speaking about $\mathcal{S}$-semicontinuity, we shall mean the $\mathcal{S}$-semicontinuity of a multifunction $F\colon X \to Y$ with respect to a topology on Y induced by the uniformity $\mathscr{V}$.

It is known that there exists a system $\mathscr{P}$ of pseudometrics such that $\mathscr{B} = \{\{(x, y) \in Y \times Y\colon p(x, y) < 1\}\colon p \in \mathscr{P}\}$ is a base for $\mathscr{V}$ and $\mathscr{P}$ is the smallest system in the sense of cardinality. When $p \in \mathscr{P}$, $r \in R^+$ and $y \in Y$ denote

$$V_{p,r}[y] = \{Z \in Y\colon p(y, z) < r\}$$

and when $A \subset Y$ denote

$$V_{p,r}[A] = \bigcup_{y \in A} V_{p,r}[y].$$

Using the system $\mathscr{P}$ it is possible to introduce the $\mathscr{S}$-semicontinuity as follows. A multifunction $F\colon X \to Y$ is said to be l.$\mathscr{S}$-s.c. at a point $x \in X$ if for any $p \in \mathscr{P}$, $r \in R^+$ and $y \in F(x)$ there exists a set $A \in \mathscr{S}_x$ such that $F(z) \cap V_{p,r}[y] \neq \emptyset$ for any $z \in A$. A multifunction F is said to be u.$\mathscr{S}$-s.c. at x if $F(x)$ is compact and for any $p \in \mathscr{P}$ and $r \in R^+$ there exists a set $A \in \mathscr{S}_x$ such that $F(z) \subset V_{p,r}[F(x)]$ for any $z \in A$.

These definitions coincide with the ones introduced in Definition 5. When the compactness of $F(x)$ is omitted, then Definition 5 implies the definition introduced above but not vice-versa.

Before aproaching the main theorems we introduce another notation. Let F be a multifunction from X to Y. For any $y \in Y$ and $p \in \mathscr{P}$ define a function $f_{y,p}\colon X \to R_0^+$ as follows

$$f_{y,p}(x) = \inf\{p(y, z)\colon z \in F(x)\}.$$

For any finite set $I \subset Y$ define a function $f_{I,p}\colon X \to R_0^+$ as follows

$$f_{I,p}(x) = \min\{f_{y,p}(x)\colon y \in I\}.$$

It is evident that

$$f_{I,p}(x) = \inf\{\min\{p(y, z)\colon y \in I\}\colon z \in F(x)\}.$$

Theorem 1. A multifunction $F\colon X \to Y$ is l.$\mathscr{S}$-s.c. at a point $x_0 \in X$ if and only if the functions $f_{y,p}$ are o.u.$\mathscr{S}$-s.c. at x_0 for any $p \in \mathscr{P}$ and $y \in H$, where $\bar{H} = Y$.

Proof. Necessity. Let $r \in R^+$, $y \in H$ and $p \in \mathscr{P}$. From the property of infimum follows the existecne of $y_0 \in F(x_0)$ such that

$$p(y, y_0) < f_{y,p}(x_0) + \frac{r}{2}.$$

Since F is l.$\mathscr{S}$-s.c. at x_0, there exists a set $A \in \mathscr{S}_{x_0}$ such that

$F(x) \cap V_{p,\frac{r}{2}}[y_0] \neq \emptyset$ for any $x \in A$. Therefore there is $y_x \in F(x)$ such that $p(y_0, y_x) < \frac{r}{2}$. Hence we obtain

$$f_{y,p}(x) \leq p(y_x, y) < f_{y,p}(x_0) + r.$$

Sufficiency. Let $y \in F(x_0)$, $r \in R^+$ and $p \in \mathscr{P}$. Since H is dense in Y there is $y_0 \in H \cap V_{p,\frac{r}{4}}[y]$. The function $f_{y_0,p}$ is o.u.$\mathscr{S}$-s.c. at x_0 and therefore there is a set $A \in \mathscr{S}_{x_0}$ such that

$$f_{y_0,p}(x) < f_{y_0,p}(x_0) + \frac{r}{4}$$

for any $x \in A$.

Since $p(y, y_0) < \frac{r}{4}$, we have

$$f_{y_0,p}(x_0) < \frac{r}{4} \quad \text{and therefore} \quad f_{y_0,p}(x) < \frac{r}{2}.$$

Thus we see that $V_{p,\frac{r}{2}}[y_0] \cap F(x) \neq \emptyset$ and therefore $V_{p,r}[y] \cap F(x) \neq \emptyset$ for any $x \in A$.

In the next theorem the notion of a locally boundedness is used.

Definition 6. A multifunction $F\colon X \to Y$ is said to be localy bounded at a point $x \in X$ if there exists a neighbourhood U of x such that $\overline{F(U)} = \overline{\bigcup_{z \in U} F(z)}$ is compact. F is said to be locally bounded if it is locally bounded at any $x \in X$.

Theorem 2. Let a multifunction $F\colon X \to Y$ be locally bounded at $x_0 \in X$ and $F(x_0)$ be compact. Then F is u.$\mathscr{S}$-s.c. at x_0 if and only if all functions $f_{I,p}$ are o.l.$\mathscr{S}$-s.c. at x_0 for any $p \in \mathscr{P}$ and for any finite set $I \subset H$, where $\bar{H} = Y$.

Proof. Necessity. Let $p \in \mathscr{P}$, $r \in R^+$, $\bar{H} = Y$, $I \subset H$ and I be finite. Since F is u.$\mathscr{S}$-s.c., there exists a set $A \in \mathscr{S}_{x_0}$ such that $F(x) \subset V_{p,\frac{r}{2}}[F(x_0)]$ for any $x \in A$. From the definition of $f_{I,p}$ it follows that for any $x \in A$ there exists $y_x \in F(x)$ such that

$$\min\{p(y, y_x)\colon y \in I\} < f_{I,p}(x) + \frac{r}{2}.$$

Furthermore, we can find $z_x \in F(x_0)$ satisfying $p(z_x, y_x) < \frac{r}{2}$ and since $f_{I,p}(x_0) \leq \min\{p(y, z_x)\colon y \in I\}$ we obtain

$$f_{I,p}(x_0) \leq \min\{p(y, y_x)\colon y \in I\} + p(y_x, z_x) < f_{I,p}(x) + r.$$

Sufficiency. Let $p \in \mathscr{P}$, $r \in R^+$. Since$F$ is locally bounded at x_0, there exists a neighbourhood U of x_0 such that $\overline{F(U)} = K$ is compact. Denote $M = K \setminus V_{p,r}[F(x_0)]$. M is totally bounded, being a subset of the compact set K. It means there are $y_1, \ldots, y_n \in M$ such that $M \subset \bigcup_{m=1}^{n} V_{p,\frac{r}{4}}[y_m]$. From the density of H it follows that for any $m = 1, \ldots, n$ there exists $\bar{y}_m \in H$ such that $\bar{y}_m \in V_{p,\frac{r}{4}}[y_m]$. Thus $M \subset \bigcup_{m=1}^{n} V_{p,\frac{r}{2}}[\bar{y}_m]$.

Denote $I = \{\bar{y}_1, \ldots, \bar{y}_n\}$. Since for any $m = 1, \ldots, n$

$$f_{\bar{y}_m,p}(x_0) = \inf\{p(\bar{y}_m, y) \colon y \in F(x_0)\} > \frac{3r}{4}$$

holds, we have $f_{I,p}(x_0) > \frac{3r}{4}$. Since $f_{I,p}$ is o.l.$\mathscr{S}$-s.c. at x_0 there is a set $A \in \mathscr{S}_{x_0}$ such that

$$f_{I,p}(x) > f_{I,p}(x_0) - \frac{r}{4} > \frac{r}{2}$$

for any $x \in A \in$.

Then $f_{\bar{y}_m,p}(x) > \frac{r}{2}$ and that means $V_{p,\frac{r}{2}}[\bar{y}_m] \cap F(x) = \emptyset$ for any $m = 1, \ldots, n$. $\mathscr{S}_{x_0}$ is a stronlgy local sieve. Hence a set $A_0 = A \cap U$ belongs to $\mathscr{S}_{x_0}$ and $F(x) \subset K$ holds for any $x \in A_0$. Thus

$$F(x) \subset K \setminus \left(\bigcup_{m=1}^{n} V_{p,\frac{r}{2}}[\bar{y}_m] \right) \subset K \setminus M = K \setminus \left(K \setminus V_{p,r}[F(x_0)] \right) \subset V_{p,r}[F(x_0)]$$

and the u.$\mathscr{S}$-s.c. at x_0 is proved.

Note that if $\mathscr{S}_{x_0} = \mathscr{U}_{x_0}$, we obtain already known theorems concerning the semicontinuity. In case Y is a metric space such characterization is metioned in [1].

The following examples show that the compactness and the locally boundedness cannot be omitted in Theorem 2. In both examples $X = Y = R_0^+$ with metric $d(x, y) = |x - y|$, $\mathscr{S}_{x_0} = \mathscr{U}_{x_0}$ and $F \colon X \to (Y, d)$.

Example 1. $F(0) = \langle 0, 1)$

$F(x) = \langle 0, 1 + x \rangle$, if $x \neq 0$.

Then $f_{y,d}(x) = \max\{y - x - 1, 0\}$ for any $y \in Y$. We can see that all $f_{y,d}$ are even continuous. Therefore functions $f_{I,d}$ are continuous for any finite set $I \subset Y$, too. But F is not u.s.c. at the point 0.

Example 2. $F(x) = \{0\}$, if $x \neq \frac{1}{n}$ for $n = 1, 2, \ldots$

$$F\left(\frac{1}{n}\right) = \{n\}, \text{ for } n = 1, 2, \ldots.$$

If $I = \{y_1, \ldots, y_n\}$, then $f_{I,d}(0) = \min I = y_m$ for a certain m, where $1 \leq m \leq n$. Denote $z = \max I$. If $x < \frac{1}{2z}$, then $f_{I,d}(x) \geq y_m$. Thus all $f_{I,d}$ are o.l.s.c. at the point 0, but F is not u.s.c. at 0.

3. Points of discontinuity

Denote the set of all points at which a multifunction F is not u.$\mathscr{S}$-s.c., l.$\mathscr{S}$-s.c. by $D^+_{\mathscr{S}}F$ and $D^-_{\mathscr{S}}F$ respectively. Analogously a notation $D^+_{\mathscr{S}}f$ and $D^-_{\mathscr{S}}f$ for the function f is used.

With respect to the previous section we can see that

$$D^-_{\mathscr{S}}F = \cup \{D^+_{\mathscr{S}}f_{y,p}: y \in H, p \in \mathscr{P}\}$$

and when F is compact-valued and locally bounded, then

$$D^+_{\mathscr{S}}F = \cup \{D^-_{\mathscr{S}}f_{I,p}: I \subset H, I \text{ finite}, p \in \mathscr{P}\}.$$

If the index $\mathscr{S}$ is omitted we obtain equalities concerning usual semicontinuity. A multifunction F is said to be continuous at a point x if it is both upper and lower semicontinuous at x. Thus the set of the points of discontinuity of F is equal to $D^+F \cup D^-F$. The symbol $DF(Df)$ for the set of the points of discontinuity of a multifunction F (a function f) is used.

By the term of a monotone multifunction with domain R we understand an increasing or decreasing multifunction in the following sense. If $x_1 < x_2$, then $F(x_1) \subset F(x_2)$ or $F(x_1) \supset F(x_2)$.

Proposition 1. Let Y be a second countable uniform space and $F: R \to Y$ be monotone, compact-valued. Then the set DF is countable.

Proof. Suppose, for instance, F to be increasing. Then $F((x-1, x+1)) \subset F(x+1)$ for any $x \in R$ and since $F(x+1)$ is compact, F is locally bounded.

According to our assumption there exists a countable set H dense in Y and the set P is countable, too. Since

$$DF = \cup \{Df_{I,p}: I \subset H, I \text{ finite}, p \in \mathscr{P}\},$$

the proof will be finished after observing that any set $Df_{I,p}$ is countable.

From the definition of $f_{I,p}$ it follows that if $x_1 < x_2$, then

$$f_{I,p}(x_1) = \min\{\inf\{p(y,z)\colon z \in F(x_1)\}\colon y \in I\} \geq$$
$$\geq \min\{\inf\{p(y,z)\colon z \in F(x_2)\}\colon y \in I\} = f_{I,p}(x_2),$$

because $F(x_1) \subset F(x_2)$. Thus $f_{I,p}\colon R \to R$ is a decreasing function and therefore $Df_{I,p}$ is countable.

The same result is given in [5] for Y being a metric space.

In the proof of the next result concerning the points of discontinuity the following three lemmas are used.

Lemma 1. Let $f\colon X \to R$ be order upper (lower) semiquasicontinuous on X and order lower (upper) almost semicontinuous at a point $x_0 \in X$ Then f is order lower (upper) semicontinuous at x_0.

A simple proof of this lemma can be omitted.

Lemma 2. If $f\colon X \to R$, then f is order upper (lower) almost semicontinuous on X except for the set of the first category.

This lemma follows directly from [6].

Lemma 3. (Theorem 2.2. in [4]). If $f\colon X \to R$. is o.u.q.c. (o.l.q.c.), then f is o.u.s.c. (o.l.s.c.) except for the set of the first categoryy.

Proposition 2. Let Y be a second countable space and $F\colon X \to Y$ be compact-valued and locally bounded. If F is u.q.c. (l.q.c.), then the set DF is of the first category.

Proof. For the same reason as in Proposition 1 we can take the sets H and $\mathscr{P}$ countable and therefore it is sufficient to show that $Df_{I,p}$ is of the first category for any finite set $I \subset H$ and for any $p \in \mathscr{P}$.

Suppose F to be u.q.c.. From Theorem 2 it follows that $f_{I,p}$ is o.l.q.c.. Combining Lemma 1 and Lemma 2 we obtain that $D^+f_{I,p}$ is of the first category. According to Lemma 3 the set $Df_{I,p} = D^+f_{I,p} \cup D^-f_{I,p}$.

The proof for F being l.q.c. is analogous.

The theorems of this type, but with the assumption of the semicontinuity of F instead of the semiquasicontinuity, can be found in [5] for Y being a metric space, and in [4], [6] for Y being a topological space.

4. Connection between continuity and $\mathscr{S}$-continuity

In this section we suppose Y is a pseudometric space with a pseudometric p, and local sieves $\mathscr{S}_x$ satisfy the following condition formulated in [2].

Definition 7. We say the sieve $\mathscr{S}_x$ has a selection property if for every descending sequence $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ of the sets belonging to $\mathscr{S}_x$ a sequence $\{U_n\}_{n=1}^{\infty}$ of neighbourhoods of x and a set $A \in \mathscr{S}_x$ exist such that $A \cap U_n \subset A_n$.

In Section 1 some examples of local sieves are introduced. All of them have selection property provided that X is a Hausdorff first countable topological space.

It is obvious that in the case when Y is a pseudometric space Theorems 1 and 2 are valid. Moreover, we can omit the indices indicating the pseudometrics from the system $\mathscr{P}$.

Proposition 3. Let X be a Hausdorff first countable topological space, a multifunction $F\colon X \to Y$ be locally bounded, and u.$\mathscr{S}$-s.c. at a point $x_0 \in X$ and $F(x_0)$ be compact. The there is a set $A \in \mathscr{S}_{x_0}$ such that the restriction F/A is u.s.c. at x_0.

Proof. Let $r \in R^+$. From the property of infimum for any x and y there exists a point $z_{x,y} \in F(x)$ such that

$$p(y, z_{x,y}) < f_y(x) + \frac{r}{2}.$$

Since F is u.$\mathscr{S}$-s.c. at x_0, there exists a set $A \in \mathscr{S}_{x_0}$ such that $F(x) \subset \subset V_{\frac{r}{2}}[F(x_0)]$ for any $x \in A$. It means $\hat{z}_{x,y} \in F(x_0)$ exists such that

$$p(z_{x,y}, \hat{z}_{x,y}) < \frac{r}{2}.$$

Then

$$p(y, \hat{z}_{x,y}) < p(y, z_{x,y}) + p(\hat{z}_{x,y}, z_{x,y}) < f_y(x) + r$$

and

$$f_y(x_0) \leq p(y, \hat{z}_{x,y}) < f_y(x) + r$$

for any $y \in Y$ and $x \in A$.

Taking successivelly $r = \frac{1}{n}$, where $n = 1, 2, \ldots$, we obtain a descending sequence $\{A_n\}_{n=1}^{\infty}$ of sets from $\mathscr{S}_{x_0}$ such that

$$f_y(x) > f_y(x_0) - \frac{1}{n}$$

for any $y \in Y$ and $x \in A$.

Since $\mathscr{S}_{x_0}$ has a selection property there exist a set $A \in \mathscr{S}_{x_0}$ and a sequence $\{U_n\}_{n=1}^{\infty}$ of neighbourhoods of x_0 such that $A \cap U_n \subset A_n$. Therefore $A \cap U_n \subset \subset A \cap A_n \subset A \cap f_y^{-1}\left(\left(f_y(x_0) - \frac{1}{n}, \infty\right)\right)$ for all $y \in Y$. Since $A \cap U_n \in \mathscr{S}_{x_0}$, $A \cap f_y^{-1}\left(\left(f_y(x_0) - \frac{1}{2}, \infty\right)\right) \in \mathscr{S}_{x_0}$ for all $y \in Y$. Thus we have that $(f_y/$

$/A)^{-1}((f_y(x_0) - r, \infty)) \in \mathscr{S}_{x_0}$ for all $y \in Y$ and all $r \in R^+$. Therefore f_y/A is o.l.s.c. at x_0. If a set $I \subset Y$ is finite, then the function f_I/A is o.l.s.c. at x_0, too. Using Theorem 2 we obtain that F/A is u.s.c. at x_0.

The paper [8] deals with a restriction of this type for a quasicontinuous multifunction. There are several examples introduced in [8] to show that an analogous result for the lower semiquasicontinuity is impossible to obtain even under very restrictive conditions. But a certain result can be obtained for the so called Hausdorff lower semicontinuity denoted by H.l.s.c..

Definition 8. A multifunction F: $X \to Y$ is said to be Hausdorff lower $\mathscr{S}$-semicontinuous H.l.$\mathscr{S}$-s.c. at a point $x_0 \in X$ if for any $r \in R^+$ there exists a set $A \in \mathscr{S}_{x_0}$ such that $F(x_0) \subset V_r[F(x)]$ forany $x \in A$.

Proposition 4. Let X be a Hausdorff first countable topological space and F: $X \to Y$ be H.l.$\mathscr{S}$-s.c. at x_0. Then there exists a set $A \in \mathscr{S}_{x_0}$ such that F/A is H.l.s.c. at x_0.

Proof. For any $y \in Y$ and n positive integer there exists a point $z_y \in F(x_0)$ such that

$$p(y, z_y) < f_y(x_0) + \frac{1}{2n}.$$

Denote $A_n = \{x \in X\colon F(x) \cap V_{\frac{1}{2n}}[F(x_0)] \neq \emptyset\}$. Then for any $y \in Y$ and $x \in A_n$ we can find a point $z_{x,y} \in F(x)$ such that

$$p(z_y, z_{x,y}) < \frac{1}{2n}.$$

Thus

$$f_y(x) \leq p(y, z_{x,y}) < f_y(x_0) + \frac{1}{n}$$

holds for any $x \in A_n$ and $y \in Y$.

Since F is H.l.$\mathscr{S}$-s.c. at x_0, we have $A_n \in \mathscr{S}_{x_0}$. As shown in the previous proposition there are $A \in \mathscr{S}_{x_0}$ and $\{U_n\}_{n=1}^{\infty}$, $U_n \in \mathscr{U}_{x_0}$ such that $A \cap U_n \subset A_n$.

Thus we have $f_y(x) < f_y(x_0) + \frac{1}{n}$ for any $x \in A \cap U_n$ and $y \in Y$. Then $f_y(x_0) = 0$ for any $y \in F(x_0)$ and therefore there exists $y_x \in F(x)$ such that $p(y, y_x) < \frac{1}{n}$.

Hence $F(x_0) \subset V_{\frac{1}{n}}[F(x)]$ for any $x \in A \cap U_n$ and F/A is H.l.s.c. at x_0.

5. Sequences of multifunctions

In this section we suppose again Y to be a uniform space and we shall study some properties of the sequences of multifunctions with values in Y. The symbol N is used to denote the set of all natural numbers.

We shall deal with the following two types of convergence.

Definition 9. A sequence $\{F_n\}_{n=1}^{\infty}$ of multifunctions is called convergent to the multifunction F if for any $p \in \mathscr{P}$, $r \in R^+$ and $x \in X$ there exists $n_0 \in N$ such that $F(x) \subset V_{p,r}[F_n(x)]$ and $F_n(x) \subset V_{p,r}[F(x)]$ for any $n \in N$, $n \geq n_0$. We denote this by $F_n \to F$.

Definition 10. A sequence $\{F_n\}_{n=1}^{\infty}$ of multifunctions is called uniformly convergent to the multifunction F if for any $p \in \mathscr{P}$ and $r \in R^+$ there exists $n_0 \in N$ such that $F(x) \subset V_{p,r}[F_n(x)]$ and $F_n(x) \subset V_{p,r}[F(x)]$ for any $x \in X$ and $n \in N$, $n \geq n_0$. We denote this by $F_n \rightrightarrows F$.

The definitions are a natural generalization of a convergence when Y is a metric space and the Hausdorff distance is considered. The Definition 10 agrees with the definition of the uniform convergence introduced in another form in [9].

Remark that notations $f_n \to f$ and $f_n \rightrightarrows f$ will be used also for functions to denote the pointwise convergence and the uniform convergence respectively.

One more notation is useful. Denote

$$p_y(z) = p(y, z)$$

and

$$p_I(z) = \min\{p_y(z)\colon y \in I\} = \min\{p(y, z)\colon z \in I\}.$$

Thus $p_y\colon Y \to R_0^+$ and $p_I\colon Y \to R_0^+$ are uniformly continuous functions on Y for any $p \in \mathscr{P}$ and any $y \in Y$ or any finite $I \subset Y$.

With the help of this notation we can define for any $n \in N$, $p \in \mathscr{P}$, $y \in Y$ and finite $I \subset Y$.

$$f_{n,y,p}(x) = \inf\{p_y(z)\colon z \in F_n(x)\},$$

$$f_{n,I,p}(x) = \inf\{p_I(z)\colon z \in F_n(x)\}.$$

The next lemma is given without the proof since a similar result is proved for the case when Y is a metric space in [7] and the proof for Y being a uniform space conatins only technical differences.

Lemma 4. Let $G_n\colon X \to Y$, $g_n\colon Y \to R$ and $G_n \rightrightarrows G$, $g_n \rightrightarrows g$, where g is a uniformly continuous function on Y. If $m_n(x) = \inf\{g_n(y)\colon y \in G_n(x)\}$ and $m(x) = \inf\{g(y)\colon y \in G(x)\}$, then $m_n \rightrightarrows m$.

This lemma has a straighforward application.

Proposition 5a. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of l.$\mathscr{S}$-s.c. multifunctions and let $F_n \rightrightarrows F$. Then the multifunction F is l. $\mathscr{S}$-s.c..

Proof. If we set $G_n = F_n$, $G = F$, $g_n = g = p_y$, $m_n = f_{n,y,p}$ and $m = f_{y,p}$, then the assumptions of the previous lemma are clearly satisfied. Thus $f_{n,y,p} \rightrightarrows f_{y,p}$ for any $y \in Y$ and $p \in \mathscr{P}$. Since $f_{n,y,p}$ are o.u.$\mathscr{S}$-s.c. functions, it is sufficient to prove that the order $\mathscr{S}$-semicontinuity is preserved under a uniform convergence and then to observe that F is l.$\mathscr{S}$-s.c., when all $f_{y,p}$ are o.u.$\mathscr{S}$-s.c..

We shall not do it, because it is very similar to the proof of the fact that a uniform convergence preserves a continuity.

An analogous proposition for the upper $\mathscr{S}$-semicontinuity is given without proof.

Proposition 5b. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of u.$\mathscr{S}$-s.c. multifunctions and let $F_n \rightrightarrows F$. Then F is u.$\mathscr{S}$-s.c., provided that F is locally bounded and compact-valued.

Some results of this type, but only for semicontinuity, are given in [7] and [9].

We conclude this paper with two Dini's theorems for multifunctions. Since these theorems are again dual in a certain way, we shall prove one of them in detail and the second one will be only formulated. Before introducing the theorem, we give some lemmas in order to make the proof of it more intelligible.

Lemma 5a. If a multifunction F is u.s.c. at a point $x_0 \in X$, then for any $p \in \mathscr{P}$ and $r \in R^+$ there exists a neighbourhood U of x_0 such that

$$f_{y,p}(x) > f_{y,p}(x_0) - r$$

for any $x \in U$ and $y \in Y$.

Proof. In the proof of Proposition 3 we obtained even more general result for a u.$\mathscr{S}$-s.c. multifunction and for Y being a pseudometric space. But it is obvious that the same result is true for any $p \in \mathscr{P}$.

Lemma 5b. If a multifunction is l.s.c. at a point x_0, then for any $p \in \mathscr{P}$ and $r \in R^+$ there exists a neighbourhood U of x_0 such that

$$f_{x,p}(x) < f_{y,p}(x_0) + r$$

for any $x \in U$ and $y \in Y$.

Lemma 6. If $F_n \to F$, then for any $x \in X$, $p \in \mathscr{P}$ and $r \in R^+$ there exists $n_0 \in N$ such that

$$f_{n,y,p}(x) < f_{y,p}(x) + r$$

for any $y \in Y$ and $n \in N$, $n \geq n_0$.

Proof. Let $x \in X$, $r \in R^+$ and $p \in \mathscr{P}$. From the definition of $f_{y,p}$ for any $y \in Y$ there exists a point $z_y \in F(x)$ such that

$$p(y, z_y) < f_{y,p}(x) + \frac{r}{2}.$$

Since $F_n \to F$, there exists $n_0 \in N$ such that $F(x) \subset V_{p,\frac{r}{2}}[F_n(x)]$ for any $n \geq n_0$. It means $z_{n,y} \in F_n(x)$ exists such that

$$p(z_y, z_{n,y}) < \frac{r}{2}.$$

Then for any $y \in Y$ and $n \geq n_0$ we obtain

$$p(y, z_{n,y}) < f_{y,p}(x) + r.$$

Since $f_{n,y,p}(x) \leq p(y, z_{n,y})$, the inequality

$$f_{n,y,p}(x) < f_{y,p}(x) + r$$

is valid for any $y \in Y$ and $n \geq n_0$.

Since the proof of the following lema differs from the proof of the classical Dini's theorem for functions only in technical details, we omit it for the sake of shortness.

Lemma 7. Let X be a compact topological space and the functions $f_{n,y,p}$ and $f_{y,p}$ satisfy the following conditions:

(i) $\forall x_0 \in X \forall p \in \mathscr{P} \forall r \in R^+ \forall n \in N \exists U_1 \in \mathscr{U}_{x_0}: \forall x \in U_1 \forall y \in Y:$
$f_{n,y,p}(x) < f_{n,y,p}(x_0) + r,$

(ii) $\forall x_0 \in Y \forall p \in \mathscr{P} \forall r \in R^+ \exists U_2 \in \mathscr{U}_{x_0}: \forall x \in U_2 \forall y \in Y:$
$f_{y,p}(x) > f_{y,p}(x_0) - r,$

(iii) $\forall p \in \mathscr{P} \forall y \in Y: f_{n,y,p} \to f_{y,p}$

and, moreover,

$\forall x \in X \forall n \in N: f_{n,y,p}(x) \geq f_{n+1,y,p}(x) \geq f_{y,p}(x),$

(iv) $\forall x \in X \forall p \in \mathscr{P} \forall r \in R^+ \exists n_0 \in N: \forall n \geq n_0 \forall y \in Y: f_{n,y,p}(x) < f_{y,p}(x) + r.$

Then $\forall p \in \mathscr{P} \forall r \in R^+ \exists n_0 \in N: \forall n \geq n_0 \forall y \in Y \forall x \in X: f_{n,y,p}(x) < f_{y,p}(x) + r.$

Proposition 6a. Let X be a compact topological space. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of l.s.c. multifunctions and let a multifunction F be u.s.c.. Let $F_n \to F$ and $F_n(x) \subset F_{n+1}(x) \subset F(x)$ for any $x \in X$. Then $F_n \rightrightarrows F$.

Proof. Since F_n are l.s.c. according to Lemma 5b, we obtain that the condition (i) in Lemma 7 is satisfied. Lemma 5a and the u.s.c. of F imply (ii) in Lemma 7. The condition (iii) follows from the fact that multifunctions F_n converge to F from below. With an application of Lemma 6 we obtain (iv) in Lemma 7. Thus the conclusion of Lemma 7 is also valid.

Let now $F_n \not\rightrightarrows F$. It means there are $p \in \mathscr{P}$ and $r \in R^+$ such that for any $n \in N$ there exist $k_n > n$, $x_n \in X$ and $y_n \in F(x_n)$ satisfying

$$F_{k_n}(x_n) \cap V_{p,r}[y_n] = \emptyset .$$

But then

$$f_{k_n, y_n, p}(x_n) \geq r \quad \text{and} \quad f_{y_n, p}(x_n) = 0 ,$$

which is contraict to Lemma 7 and the proof is finished.

Proposition 6b. Let X be a compact topological space. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of u.s.c. multifunctions and let a multifunction F be l.s.c. and comact-valued. Let $F_n \to F$ and $F_n(x) \supset F_{n+1}(x) \supset F(x)$ for any $x \in X$. Then $F_n \rightrightarrows F$.

Similar Dini-type theorems can be found in [5] for Y being a metric space.

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SÚHRN

O URČITEJ CHARAKTERIZÁCII ZOVŠEOBECNENEJ SPOJITOSTI MULTIFUNKCIÍ

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V práci sa podľa vzoru [2] zavádza pojem zovšeobecnenej polospojitosti pre multifunkcie a tá sa charakterizuje pomocou zovšeobecnenej polospojitosti reálnych funkcií patriacich do určitého systému priradeného k danej multifunkcii.

Pomocou tejto charakterizácie sa dokazujú tvrdenia o bodoch nespojitosti monotónnych aj polokvázispojitých multifunkcií. Ďalej sa skúma vzťah medzi polospojiosťou a zovšeobecnenou polospojitosťou. Dokázané sú niektoré výsledky týkajúce sa postupností multifunkcií, ako napríklad Diniho veta pre multifunkcie.

РЕЗЮМЕ

ОБ ОПРЕДЕЛЕННОЙ ХАРАКТЕРИЗАЦИИ ОБОБЩЕННОЙ НЕПРЕВЫВНОСТИ МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ

Ондрей Натэр, Братислава

В статье вводится по образцу [2] понятие обобщенной полунепрерывности для многозначных отображений и оно характеризуется при помоıци обобщенной полунепрерывности реальных функций принадлежащих к системе соответствующей этой мультифункции.

Используя эту характеризацию, доказываются утвержения о точках разрыва для монотонных и для полуквазинепрерывных многозначных отображений. Дальше изучается отношение между полунепрерывностью и обобщенной полунепрерывностью. Получаются результаты, касающиеся последовательностей, как например теорема Дини для многозначных отображений.