UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
L—LI—1987

SOME PROPERTIES OF ALMOST CONTINUOUS LINEAR RELATIONS

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1. Introduction

Linear relations were studied by R. Arens [1] and A. Száz and G. Száz [2]. In [1], [2] algebraic properties of linear relations are given and in [3], [4] some topological properties of linear relations.

In the present paper we deal with almost continuous linear relations. We prove that under some hypotheses almost continuous graph-closed linear relations are continuous.

2. Some properties of almost continuous relations

All spaces considered in this part are topological spaces. If S is a relation from X into Y (i.e. a set $S \subset X \times Y$ such that $S(x) = \{y \in Y, (x, y) \in S\} \neq \emptyset$ for all $x \in X$), then for $A \subset Y$ we denote $S^-(A) = \{x\colon S(x) \cap A \neq \emptyset\}$ and $S^+(A) = \{x\colon S(x) \subset A\}$.

Definition 2.1. A relation S from X into Y is said to be upper semicontinuous (lower semicontinuous) at a point x_0 if for any open set $V \subset Y$ such that $x_0 \in S^+(V)$ ($x_0 \in S^-(V)$),

$$x_0 \in \operatorname{Int} S^+(V) \ (x_0 \in \operatorname{Int} S^-(V))$$

Definition 2.2. A relation S from X into Y is said to be upper almost continuous (lower almost continuous) at a point x_0 if for any open set $V \subset Y$ such that $x_0 \in S^+(V)$ ($x_0 \in S^-(V)$), $x_0 \in \operatorname{Int} \overline{S^+(V)}$ ($x_0 \in \operatorname{Int} \overline{S^-(V)}$). (Int E, and $\bar{E}$ denote the interior and the closure of the set E respectively.)

Definition 2.3. A relation S from X into Y is said to be upper quasicontinuous (lower quasicontinuous) at a point x_0 if for open set V such that $x_0 \in S^+(V)$ ($x_0 \in S^-(V)$) $x_0 \in \overline{\operatorname{Int} S^+(V)}$ ($x_0 \in \overline{\operatorname{Int} S^-(V)}$).

If S is upper and lower semicontinuous at x_0 (upper and lower almost continuous at x_0, upper and lower quasicontinuous at x_0), then it is said to continuous be at x_0 (almost continuous at x_0, quasicontinuous at x_0).

If S is upper semicontinuous (lower semicontinuous, upper almost continuous, lower almost continuous, upper quasicontinuous, lower quasicontinuous) at any $x \in X$, then it is said to be upper semicontinuous (lower semicontinuous, upper almost continuous, lower almost continuous, upper quasicontinuous, lower quasicontinuous).

Definition 2.4. Let X, Y be uniform spaces with fixed uniformities $\mathscr{U}$ and $\mathscr{V}$ respectively. A relation S from X into Y is said to be uniformly lower semicontinuous (uniformly lower almost continuous) if for any $C \in \mathscr{V}$ there exists $U \in \mathscr{U}$ such that $S^-(V[y]) \supset U[x]$ ($\overline{S^-(V[y])} \supset U[x]$) for any $(x, y) \in S$. ($V[y] = \{z\colon (y, z) \in V\}$.)

Remark 2.5. It is evident that uniformly lower semicontinuity (uniformly lower almost continuity) implies lower semicontinuity (lower almost continuity) in uniform topologies.

Remark 2.6. In what follows the symbol $S\colon X \to Y$ denotes a relation from X into Y.

Remark 2.7. It is obvious that if S is upper almost continuous (lower almost continuous) at $x_0 \in X$ then for any open set $U \subset Y$ such that $x_0 \in S^+(U)$ ($x_0 \in S^-(U)$) there exists an open set $V \subset X$ such that $x \in V$ and $S^+(U)$ ($S^-(U)$) is dense in V.

Proposition 2.8. Let $S\colon X \to Y$. Let Y be a regular space. Let S be upper quasicontinuous and lower almost continuous. Then S is lower semicontinuous.

Proof. Let $x_0 \in X$. Suppose S not to be lower semicontinuous at x_0. Then there exists an open set V in Y such that $x_0 \in S^-(V)$ and $x_0 \notin \operatorname{Int} S^-(V)$. Therefore there exist a $y \in S(x_0) \cap V$ and an open set V_1 in Y such that $y \in V_1 \subset \bar{V}_1 \subset V$. Then $x_0 \in \operatorname{Int} \overline{S^-(V_1)}$ since S is lower almost continuous at x_0 and $x_0 \in S^-(V_1)$. That implies the existence of a $z \in \operatorname{Int} \overline{S^-(V_1)}$ for which $z \in X - S^-(V) = S^+(Y - V) \subset S^+(Y - \bar{V}_1)$. The upper continuity of S at z gives that $z \in \operatorname{Int} S^+(Y - \bar{V}_1)$. So we have $S(\operatorname{Int} S^+(Y - \bar{V}_1)) \subset Y - \bar{V}_1$ and that implies $\operatorname{Int} S^+(Y - \bar{V}_1) \cap S^-(\bar{V}_1) = \emptyset$. But, that is a contradiction since $z \in \operatorname{Int} S^+(Y - \bar{V}_1)$, $z \in \operatorname{Int} \overline{S^-(\bar{V}_1)}$ and $\operatorname{Int} S^+(Y - \bar{V}_1) \cap \overline{S^-(\bar{V}_1)} \subset \overline{\operatorname{Int} S^+(Y - \bar{V}_1) \cap S^-(\bar{V}_1)} = \emptyset$.

Corollary 2.9. (See [6]) Let $S\colon X \to Y$. Let Y be a regular space. Let S be upper semicontinuous and lower almost continuous. Then S is lower semicontinuous.

Corollary 2.10. Let $f\colon X \to Y$ be a single-valued function. Let Y be a regular space. Let f be quasicontinous and almost continuous. Then f is continuous.

Remark 2.11. Proposition 2.8. is not valid if we omit the asumption of regularity of Y.

Example 2.12. Let $X = [0, 1]$ with the usual topology $\mathcal{O}$. Let $Y = [0, 1]$. Let $\mathcal{G} = \left\{A: A \subset Y, A \in \mathcal{O} \text{ or } A = G - \left\{\frac{1}{n}\right\}_{n=1}^{\infty}, G \in \mathcal{O}\right\}$. $\mathcal{G}$ is a base of some topology $\mathcal{T}$ on Y. Y with topology $\mathcal{T}$ is not regular.

The identity from X into Y is quasicontinuous and almost continuous, but is not continuous.

Proposition 2.13. Let $S: X \to Y$. Let Y be a normal space. Let S be lower quasicontinuous on X and upper almost continuous at x_0. Let $S(x_0)$ be a closed set in Y. Then S is upper semicontinuous at x_0.

Proof. Suppose S not to be upper semicontinuous at x_0. Then there exists an open set V such that $x_0 \in S^+(V)$ and $x_0 \notin \operatorname{Int} S^+(V)$. Since Y is a normal space and $S(x_0) \subset V$, there exists an open set V_1 in Y such that $S(x_0) \subset V_1 \subset \bar{V}_1 \subset V$. The upper almost continuity of S at x_0 and $S(x_0) \subset V_1$ imply that $x_0 \in \operatorname{Int} \overline{S^+(V_1)}$. Since $x_0 \notin \operatorname{Int} S^+(V)$ and $S^+(V_1) \subset S^+(V)$, there exists a $z \in \operatorname{Int} \overline{S^-(V_1)}$ such that $z \notin S^+(V) = X - S^-(Y - V)$. That implies that $z \in S^-(Y - \bar{V}_1)$ since $S^-(Y - V) \subset S^-(Y - \bar{V}_1)$. From the lower continuity of S at z we have $z \in \operatorname{Int}$-$S^-(Y - \bar{V}_1)$. So we have proved that $z \in \operatorname{Int} S^+(\bar{V}_1) \cap \operatorname{Int} S^-(Y - \bar{V}_1)$. But, that is contradiction, because $(\operatorname{Int} S^-(Y - V_1) \cap S^+(\bar{V}_1)) \subset (X - S^+(\bar{V}_1)) \cap S^+(V_1) \subset$ $\subset (X - S^+(\bar{V}_1)) \cap (\bar{V}_1) = \emptyset$.

Remark 2.14. Proposition 2.13. remains valid if Y is regular and $S(x_0)$ is compact.

Proposition 2.15. Let X, Y be uniform spaces. Let Y be a complete pseudometric space. Let $S: X \to Y$. Let $G(S) = \{(x, y): y \in S(x)\}$ be a closed set in $X \times Y$. Let S be uniformly lower almost continuous. Then S is uniformly lower semicontinuous.

Proposition 2.15. is a consequence of Lema 6.36. in [8].

Corollary 2.16. Let X, Y be uniform spaces. Let Y be a complete pseudometric space. Let $S: X \to Y$ be an almost continuous graph-closed relation. Let S be uniformly lower almost continuous. Then S is continuous.

Proof. By Proposition 2.15. S is lower semicontinuous and by Proposition 2.13. S is upper semicontinuous, that is, S is continuous. (Y is normal. $S(x)$ is a closed set for any $x \in X$. 6.A in [8].)

3. Almost continuous linear relations

Let X and Y be vector spaces over a field K ($K = R$ or $K = C$).

Definition 3.1. A relation S from X into Y is said to be linear if $S(x) + S(y) \subset S(x + y)$ and $\lambda S(x) \subset S(\lambda x)$ for all $x, y \in X$ and $\lambda \in K$.

Definition 3.2. A function f defined on the domain of a relation S is called a selection for S if $f \subset S$.

Proposition 3.3. (See [2].) Let S: $X \to Y$ be a linear relation. Then $S(0) = \{y \in Y\colon (0, y) \in S\}$ is a vector subspace of Y and $S(x+y) = S(x) + S(y)$, $S(\lambda x) = \lambda S(x)$ for all x, $y \in X$ and $0 \neq \lambda \in K$.

Proposition 3.4. (See [2].) Let S: $X \to Y$ be a linear relation and f be a selection for S. Then $S(x) = f(x) + S(0)$ for all $x \in X$. If $S(x) \cap S(y) \neq \emptyset$, then $S(x) = S(y)$.

Proposition 3.5. (See [2].) Let S: $X \to Y$ be a linear relation. Then there exists a linear selcetion f for S.

In what follows X and Y are topological vector spaces.

Proposition 3.6. Let S: $X \to Y$ be a linear relation. Let $x_0 \in X$. Let S be upper almost continuous (lower almost continuous) at x_0. Then S is upper almost continuous (lower almost continuous).

Proof. We prove the case of upper almost continuity, the other case being similar.

First we prove that S is upper almost continuous at 0. Let U be an open set such that $S(0) \subset U$. Let $y_0 \in S(x_0)$. Then $S(x_0) \subset y_0 + U$.

Upper almost continuity at x_0 implies that $x_0 \in \operatorname{Int} \overline{S^+(y_0 + U)}$, i.e. $\operatorname{Int} \overline{S^+(y_0 + U)} - x_0$ is an open neighbourhood of 0. We prove that $S^+(U + y_0) - x_0 \subset S^+(U)$.

Let $v \in S^+(U + y_0) - x_0$. Then $v = a - x_0$ and $S(a) \subset U + y_0$, i.e. $S(v) = S(a) - S(x_0) = S(a) - y_0 + S(0) = S(a) - y_0 \subset U + y_0 - y_0 = U$, also $v \in S^+(U)$. $S^+(U+y_0) - x_0 \subset S^+(U)$ implies $\operatorname{Int}\overline{(S^+(U+y_0) - x_0)} \subset \operatorname{Int} \overline{S^+(U)}$. Since $\operatorname{Int}\overline{(S^+(U + y_0) - x_0)} = \operatorname{Int} \overline{S^+(U + y_0)} - x_0$, $\operatorname{Int} \overline{S^+(U)}$ is a neighbourhood of 0. To prove that upper almost continuity at 0 implies upper almost continuity at point $x \in X$ is analogous.

Proposition 3.7. Let S: $X \to Y$ be a linear relation. Let S be upper almost continuous. Then S is lower almost continuous.

Proof. By Proposition 3.6. it suffices to prove that S is lower almost continuous at 0.

Let $V \subset Y$ be an open set such that $S(0) \cap V \neq \emptyset$. The set $S(0) + V$ is open in Y and $S(0) \subset S(0) + V$. (Let $y \in S(0)$. Let $y_0 \in S(0) \cap V$. Then $y = (y - y_0) + y_0$, where $y_0 \in V$ and $y - y_0 \in S(0) - S(0) \subset S(0)$, i.e. $y \in S(0) + V$).

The upper almost continuity of S at 0 implies that $0 \in \operatorname{Int} \overline{S^+(S(0) + V)}$. For each x of $S^+(S(0) + V)$ we have $S(x) \cap V \neq \emptyset$, i.e. $x \in S^-(V)$, since $y - a \in S(x) \cap V$ if $y \in S(x)$, $y = a + v$, where $a \in S(0)$ and $v \in V$. Thus we have proved that $S^+(S(0) + V) \subset S^-(V)$. This implies that $0 \in \operatorname{Int} \overline{S^+(S(0) + V} \subset \operatorname{Int} \overline{S^-(V)}$ and the lower amost continuity of $S \in 0$.

Now we prove theorems 3.8. and 3.11. These theorems are given in [10] without the proof.

Theorem 3.8. Let S: $X \to Y$ be a linear relation. Let X be a space of the second category. The S is lower almost continuous.

Proof. Let 0_y be a zero in X and 0_x be a zero in Y. First we prove that $0_x \in \operatorname{Int} \overline{S^-(U)}$ for any neighbourhood U of 0_y. There exists a neighbourhood V of 0_y such that $V + V \subset U$ and $\lambda V \subset V$ for any λ with $|\lambda| \leqq 1$. $Y = \bigcup_{n=1}^{\infty} nV$, i.e., $X = S^-(Y) = \bigcup_{n=1}^{\infty} nS^-(V)$. Since X is a space of the second category, $\operatorname{Int} \overline{S^-(V)} \neq \emptyset$. Let $y \in \operatorname{Int} \overline{S^-(V)}$. Then $\operatorname{Int} \overline{S^-(V)} - y$ is an open neighbourhood of 0_x.

Let $v \in S(y)$. Then $S^-(V) - y \subset S^-(V) - S^-(\{v\}) \subset S^-(V - v)$. Since V is a neighbourhood of 0_y, there exists $\lambda > 0$ such that $v \in \lambda V$. Hence $S^-(V - v) \subset$ $\subset S^-(V - \lambda V) \subset S^-((\lambda + 1)(V - V)) \subset S^-((\lambda + 1)U) \subset (\lambda + 1)S^-(U)$. Since $\operatorname{Int} \overline{S^-(V)} - y = \operatorname{Int} \overline{(S^-(V) - y)}$ and $S^-(V) - y \subset (\lambda + 1)S^-(U)$, $0_x \in$ $\in \operatorname{Int} \overline{(\lambda + 1)S^-(U)}$, i.e. $0_x \in \operatorname{Int} \overline{S^-(U)}$. Now we prove that S is lower almost continuous. By Proposition 3.6. it suffices to prove that .S is lower almost continuous at 0_x. Let $G \subset Y$ be an open set such that $S(0_x) \cap G \neq \emptyset$. Let $y_0 \in S(0_x) \cap G$. $G - y_0$ is a neighbourhood of 0_x. Hence $0_x \in \operatorname{Int} \overline{S^-(G - y_0)}$. Since $y_0 \in S(0_x)$, $S^-(G - y_0) \subset S^-(G)$ and $0_x \in \operatorname{Int} \overline{S^-(G)}$.

Remark 3.9. The assumption on X is essential.

Example 2.10. Let $C = C\,[0, 1]$ denote the set of all real-valued continuous functions f on the interval $[0, 1]$ and define $\varrho\,(f, g) = \sup_{0 \leqq x \leqq 1} |f(x) - g(x)|$. The space is a complete metric space.

Next consider the same set C, but take for metric function $\sigma(f, g) =$ $= \int_0^1 |f(x) - g(x)|\,dx$. The space (C, σ) is of first category in itself.

Let $X = (C, \sigma)$ and $Y = (C, \varrho)$. Let I be the identity from X onto Y. Then I is linear relation, but I is not almost continuous.

(Let $U = \{f \in Y$: $\varrho(f, 0) < 1\}$. For any neighbourhood V of 0 in X there exists a nonempty open set $G \subset V$ such that $G \cap I^-(U) = \emptyset$. Let V be a neighbourhood of 0 in X. There exists $\varepsilon > 0$ such that $\{f \in X$: $\sigma(f, 0) < \varepsilon\} \subset V$. Let

$$g(x) = \begin{cases} 2 & \text{on } \left[0, \frac{\varepsilon}{4}\right] \\ -\frac{8}{\varepsilon}x + 4 & \text{on } \left[\frac{\varepsilon}{4}, \frac{\varepsilon}{2}\right] \\ 0 & \text{on } \left[\frac{\varepsilon}{2}, 1\right]. \end{cases}$$

If $|f| < 1$ then $\int_0^1 |g(x) - f(x)|\,dx \geqq \int_0^{\varepsilon/4} |g(x) - f(x)|\,dx \geqq \int_0^{\varepsilon/4} (|g(x)| -$

$- |f(x)|\,dx = \int_0^{\varepsilon/4} |g(x)|\,dx - \int_0^{\varepsilon/4} |f(x)|\,dx > 2.\frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{4}$. Hence no element of the $\frac{\varepsilon}{4}$ — neighbourhood of g in X belongs to $I^-(U)$).

Theorem 3.11. Let S: $X \to Y$ be a linear relation. Let X and Y be locally convex spaces. Let X be a barrelled space ([7]). The S is lower almost continuous.

Proof. It suffices to prove that $0_x \in \operatorname{Int} \overline{S^-(U)}$ for any neighbourhood U of 0_y. Let U be a neighbourhood of 0_y. There exists a base $\mathscr{U}$ of absolutely convex absorbing neighbouhroods of 0_y. Hence there exists $V \in \mathscr{U}$ such that $V \subset U$. Linearity of S implies that $S^-(V)$ is an absolutely convex absorbing set. Hence $\overline{S^-(V)}$ is closed absoltely convex absorbing set in X. X is a barrelled space, i.e. $\overline{S^-(V)}$ is a neighbourhood of 0_x. So $0_x \in \operatorname{Int} \overline{S^-(V)} \subset \operatorname{Int} \overline{S^-(U)}$.

Theorem 3.12. Let S: $X \to Y$ be a linear relation. Let Y be a normed vector space. Let $S(0)$ be a closed set in Y and $S(0) \neq \{0\}$. Then S is upper almost continuous if and only if the set $\{x \in X: S(x) = S(0)\}$ is dense in X.

Proof. Let the set $\{x \in X: S(x) = S(0)\}$ be dense in X. We prove that S is upper almost continuous. It sufices to prove that S is upper almost continuous at 0.

Let V be an open set in Y such that $S(0) \subset V$. Since $\{x \in X: S(x) = S(0)\} \subset S^+(V)$ and $X = \overline{\{x \in X: S(x) = S(0)\}}$, $X \subset \overline{S^+(V)}$, i.e. S is upper almost continuous at 0.

Let S be upper almost continuous. We prove that the set $\{x \in X: S(x) = S(0)\}$ is dense in X.

First we prove that there exists an open set $V \subset Y$ such that $S(0) \subset V$ and $(y + S(0)) \cap (Y - V) \neq \emptyset$ for any $y \notin S(0)$.

Put $V = \bigcup_{u \in S(O)} B(u, e^{-\|u\|^2})$, where $B(x, r) = \{y \in Y: \|x - y\| < r\}$ ($\|x\|$ denotes the norm of x in Y). V is an open set in Y and $S(0) \subset V$.

Let $x \notin S(0)$. Denote $a = \inf_{u \in S(0)} \|x - u\|$.

If $a \geqq 1$, then $x \notin V$ and so $(x + S(0)) \cap (Y - V) \neq \emptyset$.
Let $a < 1$. Take $u \in S(0)$ such that $\|u\| > \sqrt{-\ln a} + 1 + \|x\|$.
Hence $\|u + x\| \geqq \|u\| - \|x\| > \sqrt{-\ln a} + 1$. We prove that $u + x \notin V$. Suppose the contrary. Then there exists $v \in S(0)$ such that $u + x \in B(v, e^{-\|v\|^2})$,
i.e. $\|u + x - v\| < e^{-\|v\|^2}$. Since $(v - u) \in S(0)$ $\|u + x - v\| \geqq a$. Hence $a < e^{-\|v\|^2}$, i.e. $\|v\| < \sqrt{-\ln a}$.
Then $\|u + x\| = \|u + x - v + v\| \leqq \|u + x - v\| + \|v\| < e^{-\|v\|^2} + \sqrt{-\ln a} < 1 + \sqrt{-\ln a}$, contrary to the hypothesis. Take the set V. Then $\{x \in X: S(x) = S(0)\} = S^+(V)$. Let $x_0 \in S^+(V)$ and let $S(x_0) = S(0)$. Then

$S(x_0) \cap S(0) = \emptyset$ and there exists a $y_0 \in S(x_0) - S(0)$. Since $y_0 \notin S(0)$, then $S(x_0) - V = (y_0 + S(0)) \cap (Y - V) \neq \emptyset$. So we have $x_0 \in S^+(V)$.) $S^+(V)$ is the vector subspace of X. Upper almost continuity of S at 0 implies that $0 \in \operatorname{Int} \overline{S^+(V)}$. Since $\overline{S^+(V)}$ is a vector subspace of X, $\operatorname{Int} \overline{S^+(V)}$ is also a vector subspace of X. For each $x \in X$ there exists $\lambda > 0$ such that $x \in \lambda \operatorname{Int} \overline{S^+(V)} = \operatorname{Int} \overline{S^+(V)}$. Thereofere $X = \operatorname{Int} \overline{S^+(V)} = \overline{\{x \in X: S(x) = S(0)\}}$.

Remark 3.13. The assumption that Y is a normed vector space may not be omitted.

Example 3.14. Consider the set $B = B[a, b]$ of all real-valued bounded functions f on the interval $[a, b]$, and define $p_1(f) = \sup_{a \leqq x \leqq b} |f(x)|$. It is evident that (B, p_1) is a normed vector space. Next consider the set $L = L[a, b]$ of Lebesgue-integrable functions on $[a, b]$, and define $p_2(f) = \int_a^b |f(t)| \, dt$. Then p_2 is semi-norm, but p_2 is not a norm.

Let $X = \{f: [a, b] \to R, f$ being a bounded Lebesgue-integrable function$\}$ be a subspace of (B, p_1) and let $Y = (L, p_2)$.

Define $S: X \to Y$ as follows: $S(x) = x + C$, where $C = \left\{f \in Y: \int_a^b |f(t)| \, dt = 0\right\}$.

It is easy to verify that $S(0) = C$ is a closed set in Y and S is an upper semicontinuous linear relation, i.e. S is upper almost continuous. (Let V be an open set in Y such that $S(0) \subset V$. There exists $\varepsilon > 0$ such that $\{f \in Y: p_2(f) < \varepsilon\} \subset V$. Then $\left\{f \in X: p_1(f) < \dfrac{\varepsilon}{b - a}\right\} \subset S^+(V)$.) But $\{x \in X: S(x) = S(0) = C\} = \{x \in X: x \in C\}$ is not dense in X.

Corollary 3.15. Let $S: X \to Y$ be a linear relation. Let Y be a normed vector space. Let $S(0)$ be a closed set in Y and $S(0) \neq \{0\}$. Then S is upper semicontinuous if and only if $S(x) = S(0)$ for every $x \in X$.

Corollary 3.16. Let $S: X \to Y$ be a linear relation. Let Y be a normed vector space and $S(0) \neq \{0\}$. Let $G(S) = \{(x, y): y \in S(x)\}$ be a closed set in $X \times Y$. If S is upper almost continuous, then S is upper semicontinuous.

Proof. Put $L = \{x \in X: S(x) = S(0)\}$. L is a closed set in X, since $G(S)$ is a closed set in $X \times Y$. The upper almost continuity of S and Theorem 3.12 imply that L is dense in X. Hence $L = X$.

Proposition 3.17. Let $S: X \to Y$ be a linear relation. Let $f: X \to Y$ be an almost continuous selection for S. Then S is lower almost continuous.

Proof. It suffices to prove that S is lower almost continuous at 0. Let U be an open set in Y such that $S(0) \cap U \neq \emptyset$. By Proposition 3.4 $S(x) = f(x) + S(0)$ for every $x \in X$. There exists $v \in S(0)$ such that $(f(0) + v) \in U$. Almost continuity of f implies that $0 \in \operatorname{Int} \overline{f^{-1}(U - v)}$. It holds $f^{-1}(U - v) \subset S^-(U)$ since for each $x \in f^{-1}(U - V)$ we have $f(x) + v \in (f(x) + S(0)) \cap U = S(x) \cap U$. So $0 \in \operatorname{Int}$-

$\overline{S^{-}(U)}$ and S is lower almost continuous at 0. By Proposition 3.6 S is lower almost continuous.

Remark 3.18. Let S: $X \to Y$ be a linear relation. Let S fulfil the assumptions of Proposition 3.17. Then S need not be upper almost continuous. S: $R^2 \to R^2$, $S(x, y) = (x, y) + S(0, 0)$, where $S(0, 0) = \{(x, 0), x \in R\}$. $f(x, y) = (x, y)$ is a continuous selection for S but S is not upper almost continuous.

Proposition 3.19. Let S: $X \to Y$ be a linear relation. If S is lower semicontinuous (lower almost continuous), then S is uniformly lower semicontinuous (uniformly lower almost continuous) by the natural uniformities on X and Y.

Proof. We prove the case of lower semicontinuity, the other case being similar. Let $\mathscr{U}$ be the uniformity in X and $\mathscr{V}$ the uniformity in Y, S lower semicontinuous and $V \in \mathscr{V}$. There exists an open set $G \subset Y$ such that $0 \in G$, $G = -G = \{-y: y \in G\}$ and $\{(x, y): (x - y) \in G\} \subset V$. Hence $G \subset V[0]$. The lower semicontinuity of S at 0 implies that $0 \in \operatorname{Int} S^{-}(G) = \operatorname{Int} S^{-}(-G) =$ $= -\operatorname{Int} S^{-}(G)$. Let $U = \{(x, y): x - y \in \operatorname{Int} S^{-}(G)\}$. Then $U[0] \subset S^{-}(V[0])$ since for each $x \in U[0] = \operatorname{Int} S^{-}(G)$ there exists a $t \in S(x) \cap G \subset S(x) \cap V[0]$. Hence $U[x] \subset S^{-}(V[y])$ for any $(x, y) \in S$. Let $(x, y) \in S$. Then $V[y] \supset G[y] = G + y$ and $U[x] = \operatorname{Int} S^{-}(G) + x$. For each $v \in U[x]$ we have: $v = h + x$, where $h \in \operatorname{Int} S^{-}(G)$ and $S(v) = S(h) + S(x)$. Then $h \in S^{-}(G)$ implies the existence of $u \in S(h) \cap G$ and $u + y \in (S(h) + S(x)) \cap (G + y) = S(v) \cap (G + y)$. Therefore $v \in S^{-}(G + y) \cap S^{-}(V[y])$. Thus S is uniformly lower semicontinuous.

Theorem 3.20. Let S: $X \to Y$ be a linear relation. Let Y be a complete vector pseudo-metric space. Let $G(S) = \{(x, y): y \in S(x)\}$ be a closed set in $X \times Y$. Let S be lower almost continuous. Then S is uniformly lower semicontinuous.

Proof. By Proposition 3.19. S is uniformly lower almost continuous and by Proposition 2.15. S is uniformly lower semicontinuous. Theorem 3.20 can be also obtained as a consequence of a more general assertion in paper [4], [9].

Theorem 3.21. Let S: $X \to Y$ be a linear relation. Let Y be a complete vector pseudometric space. Let $G(S) = \{(x, y): y \in S(x)\}$ be a closed set in $X \times Y$. Let S be almost continuous. Then S is continuous.

Proof. By Proposition 3.19. S is uniformly lower almost continuous and by Corollary 2.16. S is continuous.

Remark 3.21. It is sufficient to suppose in 3.21 that S is upper almost continuous. (See Proposition 3.7.).

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Received: 5. 2. 1985

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SÚHRN

NEJAKÉ VLASTNOSTI SKORO SPOJITÝCH LINEÁRNYCH RELÁCIÍ

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Práca sa zaoberá skoro spojitosťou lineárnych relácií. Študuje sa zhora skoro spojitosť a zdola skoro spojitosť lineárnych relácií. Je daná charakterizácia zhora skoro spojitých lineárnych relácií pre istý typ priestorov. Zhora skoro spojitá lineárna relácia s uzavretým grafom je za istých predpokladov spojitá.

РЕЗЮМЕ

НЕКОТОРЫЕ СВОЙСТВА ПОЧТИ НЕПРЕРЫВНЫХ ЛИНЕЙНЫХ ОТНОШЕНИЙ

Люба Гола, Братислава

Эта статья занимается почти непрерывностью линейных отношений, Здесь изучается почти непрерывность сверху и снизу. В статье дана характеристика сверху почти непрерывных линейных отношений для некоторых пространств. В некоторых пространствах сверху почти непрерывное линейное отношение со замкнутым графиком непрерывное.