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REMARKS ON C-CONTIUOUS MULTIFUNCTIONS

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Introduction

C-continuous functions were introduced in [3]. Many results which are valid for the c-continuity of functions may be transferred to multifunctions. Some of them can be strengthened. On the other hand, there is a close relation between the c-continuity and a closed graph (see e.g. [8]), so some of the results for the c-continuity may be applied to the multifunctions with closed graphs. The above-mentioned questions are discussed in the present paper. The proofs of some results are straightforward and some times are the same for multifunctions as for functions. The results of this kind are stated without proofs. We include also some results which may be easily proved using some known facts (e.g. those contained in [5]). The assertions of the last type are included for the sake of completness. Another remark should be made in this introduction. Doubtlessly there are many parallel results for some types of the Θ-continuity, Θ-closed graphs and similar generalizations (see e.g. [5] for these notions). We do not follow this direction. We mean that this extension of the presented results may be left either to the reader or, if necessary, to a separate study. The main aim is to extend for multifunctions some known results about c-continuous functions and to complete some of them. Besides of the cited papers, throughout the paper there are of course many connections with the results on closed graphs of single-valued functions. From among the papers of this kind let us mention e.g. [1], [2], [6].

1. Upper and lower c-continuity

A multifunction is a mapping $F\colon K \to \mathscr{P}(Y)$, where X, Y are topological spaces and $\mathscr{P}(Y)$ the power set of Y. We write $F\colon X \to Y$ for shortness. F is said to be upper (lower) c-continuous — in symbols u.c.c (l.c.c) at $p \in X$ provided that for any open set V such that $Y - V$ is compact and $F(p) \subset V$ $(F(p) \cap V \neq \emptyset)$

there exists a neighbourhood U of p such that $F(x) \subset V$ ($F(x) \cap V \neq \emptyset$) for any $x \in U$. F is said to be u.c.c (l.c.c) if it is u.c.c (l.c.c) at any $x \in U$.

In what follows we suppose $F(x) \neq \emptyset$ for any $x \in X$.

Remark 1. If a single-valued function f: $X \to Y$ is given, then it is considered as a multifunction which associates $\{f(x)\}$ to any $x \in X$. Thus f is u.c.c (l.c.c) exactly if it is c-continuous in the sense as introduced in [3].

Remark 2. The notions of the upper (lower) c-continuity of a multifunction F: $X \to Y$ become the well-known notions of the upper (lower) semi-continuity of a multifunction if we consider on Y the topology consisting of $\emptyset$ and those open sets, the complements of which are compact (compare [7] for the standard definition of the lower and upper semi-continuity of a multifunction). We denote by u.s.c (l.s.c) the upper (lower) semi-continuity. In what follows we use for $F^+(A) = \{x\colon F(x) \subset A\}$, $F^-(A) = \{x\colon F(x) \cap A \neq \emptyset\}$, where $A \subset Y$. In case of a single-valued function f: $X \to Y$ the set $f^+(A) = f^-(A) = f^{-1}(A)$, where $f^{-1}(A)$ is the inverse image of A. Further we denote $F(E) = \bigcup_{x \in E} F(x)$.

Proposition 1. A multifunction F: $X \to Y$ is u.c.c (l.c.c) if and only if $F^+(G)$ ($F^-(G)$) is open in X for any open $G \subset Y$ such that $Y - G$ is compact.

Remark 3. It follows from the above proposition that in the case of a compact space Y the notion of u.c.c (l.c.c) coincides with that one of u.s.c (l.s.c).

Obviously the notion of u.c.c (l.c.c) is different from u.s.c (l.s.c). In fact their coincidence gives a characterization of compact spaces.

Theorem 1. A space Y is compact if and only if the following holds. For any topological space X and any multifunction F: $X \to Y$ we have: F is u.c.c (l.c.c) if and only if F is u.s.c (l.s.c).

Proof. The necessity is obvious (see Remark 3). Let us prove the sufficiency. Suppose Y not to be compact. Then there exists a net $\{x_d\}$ in Y such that it has not a convergent subnet in Y. Define X as follows: $X = Y \cup \{x_0\}$, where x_0 is an element which does not belong to Y. Let the topology on X consist of all open sets in Y and, moreover, of those sets $U \subset X$ for which $X - U$ is closed and compact in Y. Define the multifunction F: $X \to Y$ such that $F(x) = x$ if $x \in X$, $x \neq x_0$ and $F(x_0) = y_0$ where y_0 is arbitrary chosen. Evidently F is a single-valued function. Hence u.c.c and l.c.c coincide with the notion of c-continuity. The same is true for u.s.c and l.s.c. The last coincide in this case with the notion of the continuity. Since X is a compact space, there exists a subnet of the net $\{x_d\}$ converging to a point $c \in X$. Evidently $c = x_0$. In the opposite case c would be a limit of some subnet of the net $\{x_d\}$ in the space Y. But $F(x_d) = x_d$ does not converge to $F(x_0) = y_0$. So F is not continuous at x_0.

Now we prove that F is c-continuous. Obviously, it is c-continuous at any $x \neq x_0$. Let us prove the c-continuity at x_0. Take any G open in Y and containing

y_0 such that $Y - G$ is compact. Then $F^+(G) = F^-(G) = F^{-1}(G) = G \cup \{x_0\}$. But $X - (G \cup \{x_0\}) = Y - G$ is a closed and compact set. Thus $G \cup \{x_0\}$ is open in X. The c-continuity of F at x_0 is proved.

Proposition 2. Let for any compact set $C \subset Y$ the set $F^-(C)$ $(F^+(C))$ be closed. Then F is u.c.c (l.c.c).

Proof. Suppose $F^-(C)$ is closed for any compact $C \subset Y$. Take G open such that $Y - G$ is compact. Then $F^-(Y - G)$ is closed. But $F^+(G) = X - F^-(Y - G)$. So $F^+(G)$ is open. Hence F is u.c.c. The proof for l.c.c is similar.

Remark 4. The converse of Proposition 2 is not true. A simple example with single-valued function may be given (see [3]). Of course the converse of the proposition is obviously true of spaces, in which compact sets are closed. Moreover, the following characterization of such topological spaces may be obtained. Denote $\mathscr{P}$ the collection of such topological spaces in which any compact is closed. We have the following.

Theorem 2. A topological space Y belongs to $\mathscr{P}$ if and only if the following is true. For any topological space X and any multifunction $F: X \to Y$ we have F is u.c.c (l.c.c) if and only if $F^-(C)$ $(F^+(C))$ is closed for any compact $C \subset Y$.

Proof. The sufficiency follows immediately. To prove the necessity, suppose that $Y \notin \mathscr{P}$. It is sufficient to find a single-valued function $f: X \to Y$ such that f is c-continuous but $f^{-1}(C)$ is not closed for some compact set $C \subset Y$. Take $X = Y$ and C such a compact set which is not closed.

Then the identity may be taken for the function f.

Remark 5. A different characterization of $\mathscr{P}$ was given in [4].

The following two propositions are immediate. The proofs are left to the reader.

Proposition 3. If $F: X \to Y$ is u.c.c (l.c.c) and $A \subset X$, then F/A is u.c.c (l.c.c).

Proposition 4. Let $F: X \to Y$ be u.s.c (l.s.c) and let $G: Y \to Z$ be u.c.c (l.c.c). Then the composition $G \circ F$ is u.c.c (l.c.c).

We say that a multifunction $F: X \to Y$ be bounded at the point p if there exists a neighbourhood U of p and a compact set $C \subset Y$ such that $F(x) \subset C$ for any $x \in U$.

Proposition 5. Let $Y \in \mathscr{P}$. Let $F: X \to Y$ be bounded at the point p. Then F u.c.c (l.c.c) implies that F is u.s.c (l.s.c) at p.

Proof. We give the proof for u.s.c. The proof for l.s.c is similar. Let $p \in X$. Let U be open containing p and $C \subset Y$ such that $F(x) \subset C$ for any $x \in U$. Let V be open such that $F(p) \subset V$, Take $G = V \cup (Y - C)$ and denote it (1). then G is open $F(p) \subset G$, $Y - G$ is compact. So a neighbourhood W of p exists such that $F(x) \subset G$ for any $x \in W$. Put $H = W \cap U$. Then $F(x) \subset G \cap C$ for any $x \in H$. Thus in view of (1), we have $F(x) \subset V$ for any $x \in H$. The u.s.c of F at p is proved.

Corollary 1. (See [3] for the single valued functions). If F: $X \to Y$, where Y is a Hausdorf space and F is u.c.c (l.c.c) and such that there is a compact set $C \subset Y$ containing all the values $F(x)$, where $x \in X$, then F is u.s.c (l.s.c).

Proposition 6. Let X be a locally compact space and Y a Hausdorff space. Let F: $X \to Y$ be a u.c.c (l.c.c) multifunction such that $F(c)$ is a compact set for any compact $C \subset X$. Then F is u.s.c (l.s.c).

Proof. Let $x \in X$. Taking an open neighbourhood V of x such that $\bar{V}$ is compact we have that $F(\bar{V})$ is compact. So F is bounded at x and the results follow from Proposition 5.

2. Points of continuity of upper (lower) c-continuous multifunctions

Proposition 7. Let Y be a Hausdorff space. Let F: $X \to Y$ be u.c.c and such that for any $x \in X$, $F(x)$ possesses a compact neighbourhood. Then the set of all x for which F is not u.s.c is closed.

Proof. Let A be the set of all those x for which F is u.s.c. Let $x_0 \in A$. Denote W compact neighbourhood of $F(x_0)$. From the u.s.c at x_0 there is an open neighbourhood V of x_0 such that for any $x \in V$ we have $F(x) \subset W$. Hence (by Proposition 5) F is u.s.c at any $X \in V$. Thus $V \subset A$ and A is open.

The following example shows that Proposition 7 is not true if we substitute u.c.c by l.c.c.

Example 1. Let $X = \langle 0, 1\rangle$ with the topology induced by the family $\mathscr{B}$ of all half open intervals $\langle a, b)$ considered as a base. Let Y be the set af all real numbers with the usual topology. Define the multifunction F: $X \to Y$ as follows:

$$F(0) = F(1) = \{-1\}$$

$$F\left\{\frac{1}{n}\right\} = \{0, \ -1\} \text{ if } n = 2, 3, \ldots \text{ and}$$

$$F(x) = \left\{\frac{1}{x - 1/n + 1}, \ -1\right\} \text{ if } x \in (1/n + 1, \ 1/n)$$

for all positive integers.

Evidently, F is l.c.c and for any $x \in X$, $F(x)$ possesses a compact neighbourhood. The multifunction F is not l.s.c at any point $x = 1/n$, where $n \geq 2$ is an integer and the set $\left\{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ is not closed. The verification is immediate.

Corollary 2. Let Y be a Hausdorff locally compact space. Let F: $X \to Y$ be u.c.c compact-valued multifunction. Then the set of points at which F is not u.s.c is closed.

Corollary 3. Let f: $X \to Y$ be a single-valued, c-continuous function into a locally compact Hausdorff space. Then the set of all discontinuity points of f is closed.

Theorem 3. Let X be a Baire space and Y a Hausdorff space such that $Y = \bigcup_{n=1}^{\infty} C_n$, where C_n $(n = 1, 2, \dots)$ are compact sets. Let F: $X \to Y$ be a l.c.c multifunction such that for any $x \in X$ there is C_n $(n = n(x))$ with the property $F(x) \subset C_n$. Then the set of all those points for which F is not l.s.c is nowhere dense in X.

Proof. Let $U \subset X$ be any nonempty open set. U, as a subspace of X, is a Baire space. Due to Proposition 3. $F/U = F_1$ is l.c.c. The set $F^+(C_n)$ is closed for $n = 1, 2, \dots$ hence $F_1^+(C_n)$ is closed. It follows from the assumption $F(x) \subset C_n$ which is valid for any $x \in U$ and suitable $n = n(x)$ that $\bigcup_{n=1}^{\infty} F_1^+(C_n) = U$. Since U is a Baire space we have $(F_1^+(C_n))^0 \neq \emptyset$ for some n. (A^0 denotes the interior of A). Put $V = (F_1^+(C_n))^0$. We have $F_1(x) \subset C_n$ for any $x \in V$. Hence from Proposition 5 it follows that F_1 is l.s.c on V. The theorem is proved.

Corollary 4. Let X, Y be as in Theorem 3. Let f: $X \to Y$ be a single-valued, c-continuous function. Then the set of all discontuinuity points is nowhere dense in X.

Remark 6. The above corollary was proved in a more special form in [3] (see also [1]).

Remark 7. Theorem 3 is not true if we substitute l.c.c by u.c.c not changing the remaining assumption.

Example 2. Let $X = \langle 0, 1 \rangle$ with the usual topology. Let Y be the set of all positive integers with the discrete topology. The X is a Baire space and Y a Hausdorff space. Define the multifunction F: $X \to Y$ as follows:

$F(0) = F(x) = \{1\}$ if x is irrational

$F(x) = \{1, q\}$ if $x = \frac{p}{q}$ is any rational number written in the usual form.

Obviously Y may be written as $\bigcup_{n=1}^{\infty} C_n$ where $\{C_n\}_{n=1}^{\infty}$ is an increasing sequence of finite and hence compact sets. Evidently for any $x \in X$ the set $F(x) = \{1, q\}$ is a subset of suitable C_n. The multifunction F is u.c.c and not u.s.c at any $x_0 \in X$.

Theorem 4. Let X be a Baire space, Y a Hausdorff space such that $Y = \bigcup_{n=1}^{\infty} C_n$, where C_n are compact sets. Let F: $X \to Y$ be a u.c.c multifunction such that $F(F^-(C_n)) \subset C_n$ for $n = 1, 2, \dots$. Then the set of all those points at which F is not u.s.c is nowhere dense in X.

Proof. It follows from the assumption that $F^-(C_n) = F^+(C_n)$ for $n = 1, 2, \dots$. Hence the sets $F^+(C_n)$ and $F^-(C_n)$ are closed. The rest of the proof goes in a similar way as the proof of Theorem 3.

Remark 8. Many of the foregoing theorems can be proved in such formulation that the assumption on Y to be Hausdorff is substituted by the assumption that the Y belongs to $\mathscr{P}$ (see Remark 4).

3. Connections with known results and further remarks

The results are deeply connected with the results of some other autors. Many results are formally the same as those published in [5] where usualy instead of the asumption that a multifunction $F: X \to Y$ is u.c.c the author uses the assumption that F is with a subclosed graph. The resemblence will be not surprising if one takes into consideration that there is a deep connection between the upper-c-continuity and the closed graph of a multifunction. In this connection we mention the following two theorems proved in [8].

Theorem A. Let X be a topological space, Y a Hausdorff and locally compact topological space. If $F: X \to Y$ is a closed-valued upper-c-continuous multifunction, then the graph of F is closed.

Theorem B. Let X, Y be topological spaces. Let $F: X \to Y$ be a multifunction with a subclosed graph. Then F is u.c.c.

It should be noted that Theorem B follows also from Theorem 3.15. of [5] and Proposition 2.

Now from **Theorem B** it is evident that we can obtain from our results some of the results of [5] substituting the assumption that F has a subclosed graph by the assumption that F is u.c.c. Beside of the results which have been introduced, some others may be obtained for u.c.c functions. As an example we mention the following, which is in fact identical with Theorem 3. 16 of [5].

Theorem 5. Let X be a Baire space. Let Y be a Hausdorff space which is a conntable union of compact sets $\left(Y = \bigcup_{n=1}^{\infty} C_n,\text{ where } C_n \text{ are compact}\right)$. Let $\{F_t\}_{t \in T}$ be a family of a u.c.c multifunction from X to Y. Let for any $x \in X$ there is $n(x)$ such that $F_t(x) \cap C_{n(x)} \neq \emptyset$ for any $t \in T$. Then there exists an integer m and a nonempty open set V such that $F_t(x) \cap C_m \neq \emptyset$ for each $x \in V$ and $t \in T$.

To conclude the remarks let us again mention the paper [3]. Some of the results of [3] has been generalized and transferred to multifunctions. It should be mentioned that in [3] there are also results (for single-valued functions) concerning functions defined on saturated spaces.

Recall that a space X is saturated if for any $x \in X$ the intersection of all the neighbourhoods of x is a neighbourhood of x. We give here some results concerning the u.c.c continuity of multifunctions defined on such spaces.

Proposition 8. Let X be a saturated space and Y a T_1-space. Let $F: X \to Y$ be u.c.c. Then F is u.s.c.

Proof. Let $x_0 \in X$ be such that F is not u.s.c at x_0. Hence there is an open set V such that $F(x_0) \subset V$ and in any neighbourhood of x_0 there is a point z such that $F(z) \cap (Y - V) \neq \emptyset$. Denote U the intersection of all neighbourhoods of the point x_0. Then there is $z_1 \in U$ with the property $F(z_1) \cap (Y - V) \neq \emptyset$. Take $y \in F(z_1) \cap (Y - V)$. The set $Y - \{y\}$ is an open set with a compact complement. Since $F(x_0) \subset Y - \{y\}$ and F is u.c.c at x_0, there exists a G of x_0 such that for any $x \in G$ we have $F(x) \subset Y - \{y\}$. It is a contradiction because $U \in G$ so $z_1 \in G$ and $F(z_1) \not\subset Y - \{y\}$.

Collorary 5. Let X be a saturated space and Y a T_1-space. Then any single-valued function f: $X \rightarrow Y$ is continuous if and only if it c-continuous.

Proposition 9. Let X be a saturated space and f: $X \rightarrow Y$ a single-valued function which is closed-valued. Then f is continuous if and only if it is c-continuous.

Proof. Analogical to that of Proposition 8.

Proposition 10. Let X be a saturated space, Y a locally compact regular space. Let F: $X \rightarrow Y$ be a u.c.c closed-valued multifunction. Then F is u.s.c.

Proof. Let $x_0 \in X$ and let F be not u.s.c at x_0. Then there exists an open set V such that $F(x_0) \subset V$ and in any neighbourhood of x_0 there is a point z with $F(z) \cap (Y - V) \neq \emptyset$. Let U be the intersection of all neighbourhoods of x_0. There is $z_1 \in U$ such that $F(z_1) \cap (Y - V) \neq \emptyset$. Take $y \in F(z_1) \cap (Y - V)$. Since $F(x_0)$ is a closed set and $y \notin F(x_0)$, there exists an open set W, $y \in W$ such that $\bar{W}$ is a compact set and $\bar{W} \subset Y - F(x_0)$. So $F(x_0) \subset Y - \bar{W}$. Since F is u.c.c at x_0 there exists an open set G with $x_0 \in G$ and for any $x \in G$ $F(x) \subset Y - \bar{W}$. This is a contradiction, since $z_1 \in G$ and $F(z_1) \cap \bar{W} \neq \emptyset$.

Corollary 6 (see [3]). Let X be a saturated space, Y a locally compact regular space. Let f: $X \rightarrow Y$ be a single-valued c-continuous function. Then f is continuous.

Remark 9. Propositions 8, 9 and 10 are not valid if we substitute the assumption u.c.c by l.c.c.

Example 3. Let $X = \{a, b\}$ with the anti-discrete topology. Let $Y = R$ with the usual topology. Define F: $X \rightarrow Y$ such that $F(a) = \langle 1, \infty)$ and $F(b) = \langle 1, \infty) \cup \{0\}$. Then F is l.c.c, X is saturated, Y locally compact regular and $T_1 \cdot F$ is closed-valued, but F is not l.s.c at b.

But the following assertion is true.

Proposition 11. Let X be a saturated space. Let $Y \in \mathcal{P}$, F: $X \rightarrow Y$ l.c.c, and let for any $x \in X$ there is a compact set C_x such that $F(x) \subset C_x$. Then F is l.s.c.

Proof. Let $x_0 \in X$ be a point such that F is not l.s.c at x_0. Hence an open set V exists such that $F(x_0) \cap V \neq \emptyset$ and in any neighbourhood of x_0 there is a point z with $F(z) \cap V = \emptyset$. So in the intersection U of all neighbourhoods of x_0 one can find a point x such that $F(x) \subset Y - V$. Take the compact set C_x such that $F(x) \subset C_x$. We have $F(x) \subset C_x - V$ and $C_x - V$ is a compact set. The set

$Y - (C_x - V)$ is open and $F(x_0) \cap (Y - (C_x - V)) \neq \emptyset$ using l.c.c at x_0 we obtain an open set G containing x_0 such that for any $z \in G$ there is $F(z) \cap (Y - (C_x - - V)) \neq \emptyset$. It is a contradiction because $x \in G$ and $F(x) \subset C_x - V$.

Remark 8. According to the previous remark the assumption concerning the existence of C_x for $x \in X$ is essential.

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SÚHRN

POZNÁMKY K c-SPOJITÝM MULTIFUNKCIÁM

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Charakterizujú sa niektoré topologické priestory pomocou c-spojitých multifunkcií. Udáva sa štruktúra množín bodov spojitosti c-spojitých multifunkcií.

РЕЗЮМЕ

ЗАМЕЧАНИЯ К C-НЕПРЕРЫВНЫМ ОТОБРАЖЕНИЯМ

ЛЮБИЦА ГОЛА — ВЛАДИМИР БАЛАЖ — ТИБОР НОЙБРУН, Братислава

В работе характеризуются некоторые топологические пространства при помощи многозначных c-непрерывных отображений. Вводится также характеризация множества точек непрерывности таких отображений.