UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XLVIII—XLIX — 1986

SOME NEW CRITERIONS FOR SEQUENCES WHICH SATISFY DUFFIN-SCHAEFFER CONJECTURE, III

OTO STRAUCH, Bratislava

1. Introduction

Duffin and Schaeffer formulated [1, p. 255] the following conjecture (abbreviated as D.S.C. in what follows):

Let $\{q_i\}$ be a one-to-one infinite sequence of positive integers and f a nonnegative real function on reals. If the series

$$\Sigma\, \varphi(q_i) f(q_i)$$

(φ stands for Euler totient function) is divergent, then for almost all u and infinitely many i the diophantine inequality

$$\left| u - \frac{x}{q_i} \right| < f(q_i)$$

has an integral solution x coprime with q_i.

Perhaps the first natural step towards the proof of the D.S.C. is to find some special classes of $\{q_i\}$ and f which fulfil it. The next step can be done in two directions:

(a) To seek functions f such that the D.S.C. holds for any sequence $\{q_i\}$. For instance, Erdös [2] proved that this is the case for $f(q) = 1/q^2$. Further similar results can be found in [3], [4]. Or,

(b) to seek sequences $\{q_i\}$ which satisfy D.S.C. for arbitrary function f (zero values are alloved for f). Six such sequences are listed in the subsequent Examples 1—6. Their proofs are based on the criteria proved in [1], [5], [6], [7].

Example 1. The sequence $\{q^i\}$ satisfies D.S.C. for every function f.

This follows from a theorem of Duffin-Schaeffer [1, p. 250] and the fact that

$$\frac{\varphi(q^i)}{q^i} = \frac{\varphi(q)}{q} \geqq c > 0$$

Example 2. The factorial sequence $\{i!\}$ satisfies D.S.C. for every function f.

To prove this use Theorem 12 of [5] and the fact that $\{i!\}$ satisfies its assumption, for

$$\frac{\varphi(i!)}{\varphi((i+1)!)} \leqq \frac{1}{\varphi(i+1)} \leqq c < 1$$

Example 3. The sequence of Fermat numbers $\{2^{2^i}+1\}$ satisfies D.S.C. for every function f.

This example follows from Theorem 7 of [6] and the fact that the Fermat numbers are coprime in pairs, as required in this theorem.

The next three examples are consequences of Theorem 6 of [7] which says that a sequence $\{q_i\}$ satisfies D.S.C. for every functions f if it has the following two properties

(i) $$\sum_{i=1}^{\infty} \frac{\log^2 q_i}{q_i^{2\varepsilon}} < +\infty$$

(ii) $$d_{ij} \leqq (q_i q_j)^{\frac{1}{2}-\varepsilon} \qquad \text{for } i \neq j$$

where $d_{ij} = (q_i, q_j)$ denotes the g.c.d. of q_i and q_j and ε is a fixed positive number.

Example 4. The Fibonacci sequence $\{F_i\}$, $F_{i+2} = F_{i+1} + F_i$, $F_1 = F_2 = 1$ satisfies D.S.C. for every function f.

This is a consequence of two following properties of Fibonacci numbers (see e.g. [8]):

(iii) $$F_i = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i\right) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^i (1+\varepsilon_i)$$

with $\varepsilon_i \to 0$,

(iv) $$(F_i, F_j) = F_{(i,j)}$$

Condition (i) is easy to verify for (iii). To see (ii) note that

$$(i, j) \leqq \min\left(i, \frac{j}{2}\right)$$

for $i < j$. Then if $i \geqq j/2$

$$\frac{j}{2} < (i+j)\left(\frac{1}{2} - \frac{1}{8}\right)$$

and consequently

$$(F_i, F_j) \leqq (F_i \cdot F_j)^{\frac{1}{2}-\frac{1}{8}} \tag{v}$$

using (iii), (iv). Similarly, if $i < j/2$ then

$$i < (i+j)\left(\frac{1}{2}-\frac{1}{8}\right)$$

and again (v).

Example 5. The sequence $\{q^i - 1\}$ satisfies D.S.C. for every function f. The proof parallels that of Example 4 because (see p. 29 of [9])

$$(q^i - 1, q^j - 1) = q^{(i,j)} - 1$$

Example 6. The sequence $\{q^i + 1\}$ satisfies D.S.C. for every function f. The proof, as in the previous example, follows from the fact that [10].

$$(q^i + 1, q^j + 1) = \begin{cases} q^{(i,j)} + 1 & \text{if } a = \dfrac{i \cdot j}{(i,j)^2} \text{ is odd} \\ 2 & \text{if } a \text{ is even and } q \text{ is odd} \\ 1 & \text{if } a \text{ and } q \text{ are even} \end{cases}$$

This paper is a direct continuation of the previous paper [7] of this series. We shall modify a criterion previously proved in Theorem 2 of [7] from which we deduce further criteria for sequences $\{q_i\}$ of positive integers which satisfy the Duffin-Schaeffer conjecture for every nonnegative real function f such that the sequence $\{f(q_i)\}$ is nonincreasing. Our aim is to prove some criteria in direction (b). So for instance, it will follow from the subsequent considerations that if the sequence $\{q_i\}$ has the property that every of its permutation satisfies one of these criteria then we obtain a partial solution of the D.S.C. in the direction (b) mentioned above.

2. Technical preparation

Let

$$\{t_{ij}\} = \left\{\frac{x}{q_i} - \frac{y}{q_j} > 0;\ i \neq j \leqq n,\ 0 < x < q_i,\ 0 < y < q_j,\ (x, q_i) = (y, q_j) = 1\right\} \tag{1}$$

i.e. the finite sequence $\{t_{ij}\}$ consists of all the distances between rational numbers x/q_i, y/q_j, $i \neq j \leqq n$, including repetitions.

In what follows $c, c_1, c_2, \ldots$ will always denote absolute positive constants with the convention that the same symbol may take different values on different occasion.

We start with the following special case of Theorem 2 of [7].

Theorem 1. A sequence $\{q_i\}$ satisfies D.S.C. for every function f for which the sequence $\{f(q_i)\}$ is nonincreasing provided given a positive real number s, the inequality

$$\left(\sum_{t_i \leqq t} 1\right)^s \leqq ct\left(\sum_{i \leqq n} \varphi(q_i)\right)^{s+1} \tag{2}$$

is true for every sufficiently large n and every $t > 0$.

The inequality (2) will be the ground of our considerations. Furthermore, the following estimate (see Theorem 3 of [6]) and the subsequent two lemmas will be of technical importance later on

$$\sum_{0 < \frac{x}{q_i} - \frac{y}{q_j} \leqq t} 1 \leqq ct\varphi(q_i)\varphi(q_j) \frac{q_{ij}(tq_{ij}d_{ij})}{\varphi(q_{ij}(tq_{ij}d_{ij}))} \tag{3}$$

where

$$d_{ij} = (q_i, q_j),\ q_{ij} = q_iq_j/d_{ij}^2,\ q_{ij}(x) = \prod_{\substack{p \mid q_{ij} \\ p \geqq x}} p \tag{4}$$

with p running over the set of prime numbers.

Lemma 1. Suppose that for every $t > 0$ it is possible to split the sequence (1) into two sequences (not necessarily the same for different t) in such a way, that one of them satisfies (2) with $s = s_1$ and the other one with $s = s_2$ (here s_1, s_2 are fixed and independet on t). Then $\{q_i\}$ satisfies D.S.C. for every such f for which the sequence $\{f(q_i)\}$ is nonincreasing.

Proof. The hypothesis of Lemma 1 imply

$$\sum_{t_i \leqq t} 1 \leqq c \cdot \frac{1}{t}\left[\left(t \sum_{i \leqq n} \varphi(q_i)\right)^{1+\frac{1}{s1}} + \left(t \sum_{i \leqq n} \varphi(q_i)\right)^{1+\frac{1}{s2}}\right] \tag{5}$$

for every $t > 0$. Suppose that $s_1 < s_2$. Then for $t \leqq 1 \Big/ \sum_{i \leqq n} \varphi(q_i)$ we have

$$\sum_{t_i \leqq t} 1 \leqq 2c \cdot \frac{1}{t}\left(t \sum_{i \leqq n} \varphi(q_i)\right)^{1+\frac{1}{s2}} \tag{6}$$

and for $t > 1 \Big/ \sum_{i \leqq n} \varphi(q_i)$ we have

$$\sum_{t_i \leqq t} 1 \leqq 2c \cdot \frac{1}{t}\left(t \sum_{i \leqq n} \varphi(q_i)\right)^{1+\frac{1}{s1}} \tag{7}$$

Split now the sequence $\{t_i\}$ into two sequences $\{t_i\}^1$ and $\{t_i\}^2$ accordingly whether

$t_i \leqq 1 \Big/ \sum_{i \leqq n} \varphi(q_i)$ or $t_i > 1 \Big/ \sum_{i \leqq n} \varphi(q_i)$. Then $\{t_{ij}\}^1$ satisfies (6) for all $t > 0$ and $\{t_{ij}\}^2$ satisfies (7) for all $t > 0$. If now Σ_0 is any sum over $\{t_{ij}\}$, then let $\Sigma_0 = \Sigma_1 + \Sigma_2$ be its corresponding decomposition over $\{t_{ij}\}^1$ and $\{t_{ij}\}^2$. If $N(\Sigma_i)$ denotes the number of summands in the sum Σ_i, then it follows from [7, (9)], (6) and (7) that

$$N(\Sigma_i) \leqq c_i \Big(\sum_{i \leqq n} \varphi(q_i) \Big) (\Sigma_i)^{\frac{1}{1+s_i}}$$

for $i = 1, 2$. Without loss of generality we can suppose $\Sigma_0 \leqq 1$*. Then since $s_1 < s_2$, the last inequality is also true for $i = 0$ provided $c_0 = 2\max\{c_1, c_2\}$ and $s_0 = s_2$. Theorem 2 of [5] finishes the proof.

Lemma 2. Let (see (4))

$$H(t) = t^\beta \frac{q_{ij}(tq_{ij}d_{ij})}{\varphi(q_{ij}(tq_{ij}d_{ij}))}$$

with $\beta > 0$. Then there exists a constant $c(\beta)$ not depending on q_{ij} and d_{ij} such that

$$H(t) \leqq c(\beta) H(t') \tag{8}$$

for every $t < t'$.

Proof. If $p_1 < p_2 < \ldots < p_r$ are the all prime divisors of q_{ij}, then the local maxima of $H(t)$ are in points t_k determined by $t_k q_{ij} d_{ij} = p_k$, $1 \leqq k \leqq r$. If t and t' are such that $t = t_s < t_k = t'$, then

$$\frac{H(t)}{H(t')} = \frac{p_s}{p_s - 1} \cdot \frac{p_{s+1}}{p_{s+1} - 1} \cdot \ldots \cdot \frac{p_{k-1}}{p_{k-1} - 1} \left(\frac{p_s}{p_k}\right)^\beta$$

Owing to Mertens' theorem this expression can be majorized by

$$c \cdot \frac{\log p_{k-1}}{\log p_{s-1}} \left(\frac{p_s}{p_k}\right)^\beta$$

where in this case p_{k-1}, p_k and p_{s-1}, p_s are consecutive prime numbers. Using to the limit

$$\frac{\log p_i}{\log i} \to 1$$

we have (8). If $t = t_s < t' < t_{s+1}$, then $H(t) \leqq 2H(t')$ and the conclusion follows.

---

* Owing to [5, Theorem 2] it is sufficient to consider only those members in $\{t_{ij}\}$ which represent the distances between neighbouring rational numbers of the form x/q_i, $0 < x < q_i$, $(x, q_i) = 1$, $i \leqq n$.

3. Main results

We shall now derive our theorems. These are based on inequality (2). Given a positive integer n, suppose that to every ordered couple $[i, j]$, $i \neq j \leqq n$ a closed interval

$$I_{ij} = \langle \underline{t}_{ij}, \bar{t}_{ij} \rangle$$

is given. Then divide the members t_i of (1) into two parts $\{t_i\}^1$, $\{t_i\}^2$. The first of them contains those positive members $\frac{x}{q_i} - \frac{y}{q_j}$ of (1) for which the given t does not belong to the coresponding I_{ij} whereas $\{t_i\}^2$ contains the remaining ones. It is clear that $\{t_i\}^1$, $\{t_i\}^2$ can depend on t. Suppose however, that the sequence $\{t_i\}^1$ satisfies (2) for every $t > 0$ with some fixed s. This is the case, for instance, if [6, Theorem 4]

$$I_{ij} = \left\langle \frac{1}{q_{ij}d_{ij}}, \frac{\log q_{ij}}{q_{ij}d_{ij}} \right\rangle$$

and $s = 1$*. Then Lemma 1 shows that it is enough to verify (2) only for the subsequence $\{t_i\}^2$. To do this we use (3). We have then

$$\sum_{\substack{t_i \leqq t \\ t_i \in \{t_i\}^2}} 1 \leqq \sum_{\substack{i \neq j \leqq n \\ t \in I_{ij}}} ct^{1-\beta} \varphi(q_i)\varphi(q_j) t^{\beta} \frac{q_{ij}(tq_{ij}d_{ij})}{\varphi(q_{ij}(tq_{ij}d_{ij}))} \leqq$$

$$\leqq \sum_{\substack{i \neq j \leqq n \\ t \in I_{ij}}} c(\beta) t^{1-\beta} \varphi(q_i)\varphi(q_j) \bar{t}_{ij}^{\beta} \frac{q_{ij}(\bar{t}_{ij}q_{ij}d_{ij})}{\varphi(q_{ij}(\bar{t}_{ij}q_{ij}d_{ij}))}$$

Here we used the Lemma 2 and the fact that $t \leqq \bar{t}_{ij}$. We proved in Lemma 1 of [6] that

$$\frac{q_{ij}(\log q_{ij})}{\varphi(q_{ij}(\log q_{ij}))} \leqq c$$

and from it we obtain for arbitrary $\bar{t}_{ij}$ with $\bar{t}_{ij} \leqq \log q_{ij}/q_{ij}d_{ij}$ that

$$\sum_{\substack{t_i \leqq t \\ t_i \in \{t_i\}^2}} 1 \leqq ct^{1-\beta} \sum_{\substack{i \neq j \leqq n \\ t \in I_{ij}}} \varphi(q_i)\varphi(q_j) \left(\frac{\log q_{ij}}{q_{ij}d_{ij}} \right)^{\beta}$$

For arbitrary η with $0 < \eta < 1$ Holder's inequality yields

$$\sum (\varphi(q_i)\varphi(q_j))^{1-\eta} (\varphi(q_i)\varphi(q_j))^{\eta} \left(\frac{\log q_{ij}}{q_{ij}d_{ij}} \right)^{\beta} \leqq$$

---

*Owing to (23) of [5] we can give into $\{t_i\}^1$ also all members $\frac{x}{q_i} - \frac{y}{q_j}$ of (1) for which $f(q_{ij})/q_{ij} \geqq c > 0$ where c is a choose constant.

$$\leqq (\Sigma\, \varphi(q_i))^{2(1-\eta)} \cdot \left[\sum \varphi(q_i)\varphi(q_j) \left(\frac{\log q_{ij}}{q_{ij} d_{ij}} \right)^{\frac{\beta}{\eta}} \right]^{\eta} \leqq$$

$$\leqq (\Sigma\, \varphi(q_i))^{2(1-\eta)} \cdot \left[\sum \frac{1}{q_{ij}^{\frac{\beta}{\eta}-1-\delta}} \frac{1}{d_{ij}^{\frac{\beta}{\eta}-2}} \right]^{\eta} \tag{9}$$

where in the last inequality we made use of the following facts

$$q_i q_j = q_{ij} d_{ij}^2, \quad \frac{\varphi(q_i)}{q_i} \frac{\varphi(q_j)}{q_j} \leqq 1, \quad \left(\frac{\log q_{ij}}{q_{ij}} \right)^{\frac{\beta}{\eta}} \leqq c \frac{1}{q_{ij}^{\frac{\beta}{\eta}-\delta}}$$

Here $\delta > 0$ is arbitrarily small. Thus we have prepared the ground for basic result.

Theorem 2. Let s, β, η, δ be positive constants with

$$0 < \beta \leqq 1, \quad 0 < \eta < 1 \tag{10}$$

and δ arbitrary small. Let for every sufficiently large n and every $t > 0$ we have

$$\sum \frac{1}{q_{ij}^{\frac{\beta}{\eta}-1-\delta}} \frac{1}{d_{ij}^{\frac{\beta}{\eta}-2}} \leqq c t^{\frac{\beta}{\eta} - \frac{s-1}{\eta s}} \cdot \left(\sum_{i \leqq n} \varphi(q_i) \right)^{2 - \frac{s-1}{\eta s}} \tag{11}$$

where the sum runs over the all distinct couples $[q_{ij}, d_{ij}]$ with $i \neq j \leqq n$ and

$$\frac{1}{q_{ij} d_{ij}} \leqq t < \frac{1}{q_{ij}^{1-\delta} d_{ij}} \tag{12}$$

Then the sequence $\{q_i\}$ satisfies D.S.C. for every function f with the property that the sequence $\{f(q_i)\}$ is nonincreasing.

The proof follows almost immediately from (9) using Theorem 1 if we note that the number of couples $[i, j]$, $i \neq j$ for which the pairs $[q_{ij}, d_{ij}]$ are equal does not excend $2^{v(q_{ij})}$ (where $v(q_{ij})$ denotes the number of distinct prime divisors of q_{ij}) and the embody this factor into the constant δ in $1/q_{ij}^{\frac{\beta}{\eta}-1-\delta}$.

Having established this general theorem we shall now derive a variety of consequences.

A. Suppose first that

$$\frac{\beta}{\eta} = \frac{s-1}{\eta s} = 1 + \delta$$

Then (11) becomes

$$\sum_{i \neq j \leqq n} d_{ij}^{1-\delta} \leqq c\Big(\sum_{i \leqq n} \varphi(q_i)\Big)^{1-\delta} \tag{13}$$

Holder's inequality with exponents $2/(1-\delta)$, $2/(1+\delta)$ shous that the inequality (13) is true if for every $i \neq j$ we have

$$d_{ij} \leqq c\, \frac{\sqrt{\varphi(q_i)\varphi(q_j)}}{(i\cdot j)^{\frac{1+\delta+\varepsilon}{2(1-\delta)}}} \tag{14}$$

where ε, δ are arbitrarily small positive constants. This gives.

Theorem 3. Let the sequence $\{q_i\}$ be ordered according to increasing magnitude and let the corresponding sequence $\{d_{ij}\}$ satisfies (14) for some positive ε and δ. Then $\{q_i\}$ (and trivially also any rearrangement of $\{q_i\}$) satisfies D.S.C. for every function f.

B. If we take

$$\frac{\beta}{\eta} = \frac{s-1}{\eta s}, \quad 2 > \frac{s-1}{\eta s} > 1 + \delta$$

then (11) becomes the form

$$\sum \frac{d_{ij}^{2-\frac{\beta}{\eta}}}{q_{ij}^{\frac{\beta}{\eta}-1-\delta}} \leqq c\Big(\sum_{i \leqq n} \varphi(q_i)\Big)^{2-\frac{\beta}{\eta}} \tag{15}$$

If we further impose that for every $i \neq j$

$$d_{ij} \leqq c q_{ij}^{\alpha} \tag{16}$$

with α determined by

$$2 - \frac{\beta}{\eta} = \frac{1-\delta}{1+\alpha}$$

then the left hand side of (15) can be estimated by cn^2. Since

$$\varphi(q_i) \geqq c\, \frac{q_i}{\log\log q_i}$$

we obtain from (15)

$$n \leqq c\Big(\sum_{i \leqq n} \frac{q_i}{\log\log q_i}\Big)^{\frac{1-\delta}{2(1+\alpha)}} \tag{17}$$

The function $x/\log\log x$ is increasing for sufficiently large x. Therefore the right hand side of (17) will be minimal if $\{q_i\}$ is ordered according to the magnitude.

Finally note, that the equality $q_{ij} = q_i q_j / d_{ij}^2$ implies the equivalence of (16) with

$$d_{ij} \leqq c(q_i q_j)^{\frac{1}{2} - \frac{1}{4\alpha + 2}} \tag{18}$$

Thus we arrived at the following result.

Theorem 4. Let the sequence $\{q_i\}$ be ordered according to increasing magnitude and satisfies (17). If the corresponding sequence $\{d_{ij}\}$ satisfies (16) or (18) (with α an arbitrary positive constant) then $\{q_i\}$ satisfies D.S.C. for arbitrary function f.

C. For the next theorem let β, η be such that

$$\frac{\beta}{\eta} > 2$$

Further suppose that the sequence $\{d_{ij}\}$ is sufficiently sparse in the sense that the inequality

$$\sum_{\substack{d_{ij} \geqq A,\ i \neq j \\ d_{ij} \text{ are distinct} \\ q_{ij} = \text{constant}}} \frac{1}{d_{ij}^{\frac{\beta}{\eta} - 2}} \leqq \frac{c}{A^{\frac{\beta}{\eta} - 2}} \tag{19}$$

is true for every $A > 0$. Then the left hand side of (11) can be summed in following manner

$$\sum \frac{1}{q_{ij}^{\frac{\beta}{\eta} - 1 - \delta}} \frac{1}{d_{ij}^{\frac{\beta}{\eta} - 2}} = \sum_{q_{ij} \text{ are distinct}} \frac{1}{q_{ij}^{\frac{\beta}{\eta} - 1 - \delta}} \sum_{\substack{d_{ij} \geqq 1/q_{ij}t \\ q_{ij} = \text{constant} \\ d_{ij} \text{ are distinct}}} \frac{1}{d_{ij}^{\frac{\beta}{\eta} - 2}} \leqq \sum_{q_{ij} \text{ are distinct}} \frac{c}{q_{ij}^{1 - \delta}} t^{\frac{\beta}{\eta} - 2}$$

If we take

$$2 = \frac{s - 1}{\eta s}$$

then (11) is certainly true if the series

$$\sum_{q_{ij} \text{ are distinct}} \frac{1}{q_{ij}^{1 - \delta}} \tag{20}$$

is convergent for some $\delta > 0$.

The summation condition in (19) can be dropped. Namely, it is sufficient to require only this complex of conditions:

$$d_{ij} \geqq A, \quad d_{ij} \text{ are distinct}, \ i \neq j,$$

$$\frac{q_i}{d_{ij}} = \text{const.} = c_1, \quad \frac{q_j}{d_{ij}} = \text{const.} = c_2 \quad \text{with } (c_1, c_2) = 1 \tag{21}$$

To see this note, that using sums over (21) we are able to estimate the left hand side sum of (19) in such a way that we multiply this sum by $2^{v(q_{ij})}$ and then embody this factor into the constant δ in $1/q_{ij}^{1-\delta}$. This leads to the following result.

Theorem 5. Let the sequence $\{d_{ij}\}$ be satisfying (19) for some constants β, η, $0 < \beta \leqq 1$, $0 < \eta < 1$ and for every $A > 0$*, where the dash means that the summation is over (21). Let the series (20), which is made from the sequence $\{q_{ij}\}$, be convergent for some $\delta > 0$. Then the sequence $\{q_i\}$ satisfies D.S.C. for every function f.

Note that in the summations (19) and (20) the condition $i, j \leqq n$ does not occur.

D. Let β, η, δ, δ' be positive constants such that

$$\frac{\beta}{\eta} - 1 - \delta - \delta' > 0$$

and

$$\sum_{\substack{q_{ij} \geqq A \\ q_{ij} \text{ are distinct} \\ d_{ij} = \text{constant}}} \frac{1}{q_{ij}^{\frac{\beta}{\eta} - 1 - \delta}} \leqq \frac{c}{A^{\frac{\beta}{\eta} - 1 - \delta - \delta'}} \tag{22}$$

is true for every $A > 0$. Then the left hand side of (11) can be summed in following manner

$$\sum \frac{1}{q_{ij}^{\frac{\beta}{\eta} - 1 - \delta}} \frac{1}{d_{ij}^{\frac{\beta}{\eta} - 2}} = \sum_{\substack{i \neq j \leqq n \\ d_{ij} \text{ are distinct}}} \frac{1}{d^{\frac{\beta}{\eta} - 2}} \sum_{\substack{q_{ij} \geqq 1/d_{ij}t \\ q_{ij} \text{ are distinct} \\ d_{ij} = \text{constant}}} \frac{1}{q_{ij}^{\frac{\beta}{\eta} - 1 - \delta}} \leqq$$

$$\leqq c \cdot \sum_{\substack{i \neq j \leqq n \\ d_{ij} \text{ are distinct}}} d_{ij}^{1 - \delta - \delta'} \cdot t^{\frac{\beta}{\eta} - 1 - \delta - \delta'}$$

Then (11) becomes

$$\sum_{\substack{i \neq j \leqq n \\ d_{ij} \text{ are distinct}}} d_{ij}^{1 - \delta - \delta'} \cdot t^{\frac{s-1}{\eta s} - 1 - \delta - \delta'} \leqq c\left(\sum_{i \leqq n} \varphi(q_i)\right)^{2 - \frac{s-1}{\eta s}} \tag{23}$$

The constants s, η, δ, δ' from exponents in (23) can be well choice only in a form

$$2 \geqq \frac{s-1}{\eta s} \geqq 1 + \delta + \delta'$$

* and for every c_1, c_2. The constant c from (19) is independent on c_1, c_2.

thus $\delta' < 1$. The inequality (23) we can transform in

$$t \leqq c\left(\frac{U^{2-x}}{V}\right)^{\frac{1}{x-x_0}} = F(x)$$

where

$$U = \sum_{i \leqq n} \varphi(q_i), \quad V = \sum_{\substack{i \neq j \leqq n \\ d_{ij} \text{ are distinct}}} d_{ij}^{1-\delta-\delta'}, \quad x_0 = 1 + \delta + \delta', \quad x = \frac{s-1}{\eta s}$$

The function $F(x)$ is nondecreasing or nonincreasing on the interval $(x_0, 2\rangle$ if it is true or not true a following inequality

$$1 - \delta - \delta' \geqq \frac{\log V}{\log U}$$

From it follows that the best possible choice of the constants s, η, δ, δ' are following two cases

$$\frac{s-1}{\eta s} = 1 + \delta + \delta' \quad \text{or} \quad \frac{s-1}{\eta s} = 2$$

Using the first case then (23) becomes

$$\sum_{\substack{i \neq j \leqq n \\ d_{ij} \text{ are distinct}}} d_{ij}^{1-\delta-\delta'} \leqq c\left(\sum_{i \leqq n} \varphi(q_i)\right)^{1-\delta-\delta'} \tag{24}$$

Using the second case then (23) becomes

$$\sum_{\substack{i \neq j \leqq n \\ d_{ij} \text{ are distinct}}} (d_{ij}t)^{1-\delta-\delta'} \leqq c \tag{25}$$

Further suppose that $1 - \delta - \delta' > 0$ and distinct values of the sequence $\{d_{ij}\}$ are sufficiently sparse in the sense that the inequality

$$\sum_{\substack{d_{ij} \leqq A,\, i \neq j \\ d_{ij} \text{ are distinct}}} d_{ij}^{1-\delta-\delta'} \leqq cA^{1-\delta-\delta'} \tag{26}$$

is true for every $A > 0$. By (12) $d_{ij} \leqq 1/t$, thus (25) is valid. This leads to the following result.

Theorem 6. Let the sequence $\{q_{ij}\}$ be satisfying (22) for some constants β, η, δ, δ', $0 < \beta \leqq 1$, $0 < \eta < 1$, $\delta > 0$, $\delta' > 0$, $1 - \delta - \delta' > 0$ and for every $A > 0$. Let the sequence $\{d_{ij}\}$ be satisfying (26) for every $A > 0$. Then the sequence $\{q_i\}$ satisfies D.S.C. for every function f.

Owing to Theorem 5 as the last step we prove two following theorems.

Theorem 7. Let the sequence $\{q_i\}$ be satisfying

$$\frac{q_i}{q_{i+1}} \leqq c < 1$$

for every i i.e. is lacunary. Then the sequence of squares $\{q_i^2\}$ satisfies D.S.C. for every function f.

Proof. The corresponding sequence $\{q_{ij}\}$, $\{d_{ij}\}$ for the sequence of squares $\{q_i^2\}$ are $\{q_{ij}^2\}$, $\{d_{ij}^2\}$. Thus the series (20) is convergent e.g. for $\delta = 1/4$.

To see (19) let the sequence $\{d_{ij}^2\}$ for which

$$\frac{q_i^2}{d_{ij}^2} = \text{const.} = c_1^2, \quad \frac{q_j^2}{d_{ij}^2} = \text{const.} = c_2^2$$

be ordered according to increasing magnitude

$$d_{i(1)j(1)}^2 < d_{i(2)j(2)}^2 < \dots$$

From it

$$\frac{d_{i(k)j(k)}^2}{d_{i(k+1)j(k+1)}^2} = \frac{q_{i(k)}^2/c_1^2}{q_{i(k+1)}^2/c_1^2} \leqq c^2 < 1$$

i.e. the sequence $\{d_{i(k)j(k)}^2\}$ is also lacunary and thus satisfies (19) for every $\beta/\eta > 2$. Therefore the assumptions of Theorem 5 are valid.

Theorem 8. If the corresponding sequence $\{d_{ij}\}$ of $\{q_i\}$ have only finite distinct members for $i \neq j$, then the sequence $\{q_i\}$ satisfies D.S.C. for every function f.

Proof. The assertion (19) is valid automatically.

Owing to a footnote on p. 42, it is sufficient to consider only those members in the series (20) for which $\{q_{ij}\}$ having the sufficiently large number of distinct prime divisors. Put

$$q_i' = \frac{q_i}{\max_j d_{ij}} \quad (i \neq j)$$

Then the series (20) is majorized by

$$\left(\sum \frac{1}{q_i'^{1-\delta}}\right)^2$$

and this series is convergent if the members q_i' having at least two distinct prime divisors, because the members q_i' are coprime.

REFERENCES

[1] Duffin, R. J.—Schaeffer, A. C.: Khintchine's problem in metric diophantine approximation. Duke Math. J., 8, (1941), 243—255.
[2] Erdös, P.: On the distribution of the convergents of almost all real numbers. J. Number Theory, 2, (1970), 424—441.
[3] Vaaler, J. D.: On the metric theory of diophantine approximation. Pacif. J. Math., 76, (1978), no. 2, 527—539.
[4] Viľčinskii, V. T.: Rational approximations to almost all real numbers (Russian). Vesci Akad. Navuk BSSR, Ser. Fiz.-Mat. Navuk, no. 6, (1979). 20—24, 139.
[5] Strauch, O.: Duffin-Schaeffer conjecture and some new types of real sequences, Acta Math. Univ. Comenian., 40—41, (1982), 233—265.
[6] Strauch, O.: Some new criterions for sequences which satisfy Duffin-Schaeffer conjecture, I. Acta Math. Univ. Comenian., 42—43, (1983), 87—95.
[7] Strauch, O.: Some new criterions for sequences which satisfy Duffin-Schaeffer conjecture, II. Acta Math. Univ. Comenian., 44—45, (1984), 55—65.
[8] Vorobjev, N. N.: Fibonacci numbers, Blisdell Publishing Co., New York—London, 1961.
[9] Sierpinski, W.: Elementary theory of numbers, PWN, Warszawa, 1964.
[10] Problem E2378. [1972, 906; 1973, 1059], Amer. Math. Monthly.

Author's address:
Oto Strauch
Matematický ústav SAV
Obrancov mieru 49
Bratislava
814 73

Received: 23. 1. 1984

SÚHRN

NIEKTORÉ NOVÉ KRITÉRIÁ PRE POSTUPNOSTI, KTORÉ SPĹŇAJÚ DUFFIN-SCHAEFFEROVU HYPOTÉZU, III

Oto Strauch, Bratislava

V práci je okrem iného ukázené, že ak $\{F_i\}_{i=1}^{\infty}$ je postupnosť Fibonacciho čísel, f je ľubovoľná nezáporná funkcia (môže nadobúdať aj nulové hodnoty), φ je Eulerova funkcia a

$$\sum_{i=1}^{\infty} f(F_i)\varphi(F_i) = +\infty$$

potom skoro pre všetky α má nerovnosť

$$\left|\alpha - \frac{x}{F_i}\right| < f(F_i)$$

celočíselné riešenie x pre nekonečne veľa i tak, že x, F_i sú nesúdeliteľné. Tú istú vlastnosť má i postupnosť štvorcov $\{q_i^2\}_{i=1}^{\infty}$ pre ktorú $q_i/q_{i+1} \leqq c < 1$.

РЕЗЮМЕ

НЕКОТОРЫЕ НОВЫЕ ПРИЗНАКИ ДЛЯ ПОСЛЕДОВАТЕЛЬНОСТЕЙ, УДОВЛЕТВОРЯЮЩИХ ГИПОТЕЗЕ ДАФФИН-ШАФФЕРА, III

Ото Штраух, Братислава

В работе между прочим показано, что если $\{F_i\}_{i=1}^{\infty}$ — последовательность чисел фибоnači, f — любая неотрицательная функция (она может принимать и нулевые значения), φ — функция Эйлера и

$$\sum_{i=1}^{\infty} f(F_i)\varphi(F_i) = +\infty$$

то для почти всех α неравенство

$$\left|\alpha - \frac{x}{F_i}\right| < f(F_i)$$

имеет целочисленно решение x для бесконечно многих i такое, что x, F_i — взаимно простые. Таким же свойством обладает и последовательность квадратов $\{q_i^2\}_{i=1}^{\infty}$ где $q_i/q_{i+1} \leqq c < 1$.