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THE DECOMPOSITION OF AN INTERVAL OF NATURAL NUMBERS INTO THREE STRONGLY SUM-FREE SETS

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Introduction

The set A of natural numbers is called strongly sum-free if an equation

$$x + y = z, \quad x \neq y \tag{1}$$

has no solution in the set A. In this paper we summarize the results from [1] (the Ph. D. thesis). According to [1] the longest interval $[n, m]$ of natural numbers

$$[n, m] = \{n, n + 1, n + 2, \ldots, m\}$$

divisible into three strongly sum-free sets is the interval $[n, 14n + 7]$ for $n \geqq 2$, and the interval $[1, 23]$ for $n = 1$. Hitherto known precise results are the following:

In the paper [3] it is proved that $[n, 5n + 2]$ for $n \geqq 2$ is the longest interval divisible into two strongly sum-free sets. That is a solution of P. Turán's problem. For $n = 1$ it is the interval $[1, 8]$.

If we omit from (1) the condition $x \neq y$, i.e. we look for decompositions into sum-free sets, then $[1, 13]$ is the longest interval divisible into three sum-free sets, and $[1, 44]$ is the longest interval divisible into four sum-free sets (see [4], [5]). The length of these intervals is the so-called Schur function.

In the paper [3] a decomposition introduced is of the interval $[n, 14n + 7]$ into three strongly sum-free sets

$A_0 = [n, 2n] \cup [4n + 3, 5n + 2] \cup [10n + 7, 11n + 6] \cup [13n + 8, 14n + 7]$
$B_0 = [2n + 1, 4n + 2] \cup [11n + 7, 13n + 7]$
$C_0 = [5n + 3, 10n + 6]$

If we add the number $14n + 8$ into one of these sets, then this set does not hold the property of the strongly sum-free set. The main result of the paper [1] is the following theorem which says that every other decomposition of the interval $[n, 14n + 7]$ is differ from the decomposition of A_0, B_0, C_0 only very slightly.

Theorem 1. Let A, B, C be an arbitrary decomposition of an interval $[n, 14n + 7]$ $(n \geqq 2)$ into three strongly sum-free sets with the signature where $n \in A$, and B includes the first number from $[n, 14n + 7]$ which does not belong to A (this will be presupposed in the following). Then it holds that

(i) $A \supset [n, 2n] \cup [4n + 3, 5n + 2] \cup [10n + 7, 11n + 6] \cup$
$\cup [13n + 8, 14n + 7]$
$B \supset [2n + 1, 4n + 2] \cup [11n + 7, 13n + 7]$
$C \supset [5n + 3, 7n + 3] \cup [8n + 6, 10n + 6]$

(ii) (a) $7n + 4 \in A$ and $8n + 4$, $8n + 5 \in C$, or
(b) $7n + 4 \in C$, $7n + 5 \in A$ and $8n + 5 \in C$, or
(c) $7n + 4$, $7n + 5 \in C$

(iii) All the rest numbers form the interval $[7n + 4, 8n + 5]$ can be arbitrarily placed into the sets A or C in any of the case (a), (b), (c).

It follows immediately from Theorem 1 that the number $14n + 8$ cannot be placed into any of the sets A, B, C by an arbitrary decomposition of the interval $[n, 14n + 7]$, because

$$14n + 8 = n + 13n + 8 = 2n + 1 + 12n + 7 = 6n + 8n + 8$$

and because by an arbitrary decomposition it holds that

$$n,\ 13n + 8 \in A, \quad 2n + 1,\ 12n + 7 \in B, \quad 6n,\ 8n + 8 \in C.$$

So the interval $[n, 14n + 7]$ for $n \geqq 2$ is the longest one.

Proof of Theorem 1. The complete proof is introduced in [1] and it consists of approximately 200 pages. Here we introduce the main steps of the proof only.

Let A, B, C be a decomposition of the interval $[n, 14n + 7]$ into three strongly sum-free sets. Let $n \geqq 3$. Then

1. $[n, 2n - 1] \subset A$
2. If $[n, 2n - 1] \subset A$ then $2n \in A$
3. If $[n, 2n] \subset A$ then $[2n + 1, 4n - 1] \subset B$
4. If $[n, 2n] \subset A$, $[2n + 1, 4n - 1] \subset B$ then $4n \in B$
5. If $[n, 2n] \subset A$, $[2n + 1, 4n] \subset B$ then $4n + k \notin C$ for $k = 1, 2, \ldots, n - 1$
6. If $[n, 2n] \subset A$, $[2n + 1, 4n] \subset B$, $4n + k \notin C$ $(k = 1, 2, \ldots, n - 1)$ then $4n + 1 \in B$
7. If $[n, 2n] \subset A$, $[2n + 1, 4n + 1] \subset B$, $4n + k \notin C$ $(k = 1, 2, \ldots, n - 1)$ then $4n + 2 \in B$.

Consequence.

8. $[4n + 3, 5n - 1] \subset A$

9. If $[n, 2n] \cup [4n+3, 5n-1] \subset A$, $[2n+1, 4n+2] \subset B$ then $5n \in A$
10. If $[n, 2n] \cup [4n+3, 5n] \subset A$, $[2n+1, 4n+2] \subset B$ then $5n+1 \in A$
11. If $[n, 2n] \cup [4n+3, 5n+1] \subset A$, $[2n+1, 4n+2] \subset B$ then $5n+2 \in A$

From this it follows that the decomposition of the interval $[n, 7n+2]$ into sets A, B, C ($n \geqq 3$) has to have the form

12. $A \supset [n, 2n] \cup [4n+3, 5n+2]$
$B \supset [2n+1, 4n+2]$
$C \supset [5n+3, 7n+2]$

For $n = 2$ it holds that

13. $\{2, 3, 4\} \subset A$
14. $\{5, 6, 7\} \subset B$
15. If $\{2, 3, 4\} \subset A$, $\{5, 6, 7\} \subset B$ then $8 \in B$
16. If $[2, 4] \subset A$, $[5, 8] \subset B$ then $9 \in B$
17. If $[2, 4] \subset A$, $[5, 9] \subset B$ then $10 \in B$
18. If $[2, 4] \subset A$, $[5, 10] \subset B$ then $11 \in A$
19. If $[2, 4] \subset A$, $11 \in A$, $[5, 10] \subset B$ then $12 \in A$

Then 12. holds also for $n = 2$. The following assertions hold for an arbitrary $n \geqq 2$.

20. If 12. then $7n+3 \in C$
21. If 12., $7n+3 \in C$ then $8n+3+k \notin B$ for $k = 1, 2, \ldots, 2n-3$

Consequence.

22. $[8n+7, 10n] \subset C$
23. If 12., 22. then $8n+6 \in C$
24. If 12., 22., $8n+6 \in C$ then $10n+k \in C$ for $k = 1, 2, 3, 4, 5, 6$
25. If 12., $[8n+6, 10n+6] \subset C$ then $10n+6+k \notin B$ for $k = 1, 2, \ldots, n$.

The assertions (i), (ii), (iii) from Theorem 1 follow as a consequence of the preceding statements.

Proof of (i). Hence $[5n+3, 7n+3] \subset C$ then $10n+6+k \notin C$ for $k = 1, 2, \ldots, n$ and then by 25. we have $[10n+7, 11n+6] \subset A$. Hence $[n, 2n] \cup [10n+7, 11n+6] \subset A$, $[5n+3, 7n+3] \subset C$ then $[11n+7, 13n+6] \subset B$. From the fact that $6n+4$, $7n+3 \in C$ we obtain $13n+7 \in A$ or $13n+7 \in B$. If $13n+7 \in A$, then

$$14n+7 = n + (13n+7) = 3n + (11n+7) = 6n + (8n+7)$$

by that n, $13n+7 \in A$, $3n$, $11n+7 \in B$, $6n$, $8n+7 \in C$ which is a contradiction, therefore $13n+7 \in B$. Hence

$$[5n+3, 7n+3] \cup [8n+6, 10n+7] \subset C,$$
$$[2n+1, 4n+2] \cup [11n+7, 13n+7] \subset B \text{ then}$$
$$[13n+8, 14n+7] \subset A.$$

Finally we have to find a decomposition of the interval $[7n + 4, 8n + 5]$ into the sets A, B, C mentioned in (ii), (iii).

Proof of (ii), (iii). Hence $[2n + 1, 4n + 2] \subset B$ and by 21. $8n + 3 + k \notin B$ not even one number from the interval $[7n + 4, 8n + 5]$ belongs to B, therefore $[7n + 4, 8n + 5] \subset A \cup C$. If we construct all additions $x + y$ $(x \neq y)$, where x, y are from intervals $[n, 2n]$, $[4n + 3, 5n + 2]$, $[10n + 7, 11n + 6]$, $[13n + 8, 14n + 7]$ included in A we get intervals $[2n + 1, 4n - 1]$, $[5n + 3, 7n + 2]$, $[11n + 7, 12n + 6]$, etc., if we construct all subtractions $x - y$ $(x \neq 2y)$ we get intervals $[1, n - 1]$, $[2n + 2, 4n + 2]$, $[5n + 5, 7n + 3]$, $[8n + 6, 10n + 4]$, etc. Therefore not even one number of the interval $[7n + 4, 8n + 5]$ belongs to these intervals. If we get the whole interval $[7n + 4, 8n + 5]$ into A, then the equation (1) is solvable only in the form

$$8n + 4 = (7n + 4) + n, \quad 8n + 5 = (7n + 5) + n$$

because the corresponding intervals which we get from additions of $x + y$ $(x \neq y)$, where e.g. $y \in [7n + 4, 8n + 5]$ are $[8n + 4, 10n + 5]$, $[11n + 7, 13n + 7]$, etc.; and intervals which we get from subtractions $x - y$ $(x \neq 2y)$ are $[1, n + 1]$, $[2n + 2, 4n + 2]$, $[5n + 3, 7n + 3]$.

If we get the whole interval $[7n + 4, 8n + 5]$ into C, the equation (1) has no solution in C, because in that way we get the sets A_0, B_0, C_0 from the introduction.

Entirely, using an arbitrary method we can divide the interval $[7n + 6, 8n + 3]$ into the sets A and C.

In [1] we prove the assertions 1.—25. by use of a contradiction. This method is extremely elementary. Let the sets A, B, C be a decomposition of the interval $[n, 14n + 7]$ into three strongly sum-free sets. Let us take out suitable sets $X \subset A$, $Y \subset B$, $Z \subset C$ and let us construct a triad $(X, Y, Z)_0$ from them. We can select numbers x, y from one of the sets Y, X, Z, e.g. from X and we construct $x + y$ (if $x \neq y$) or $x - y$ (if $x \geqq y + n$, $x \neq 2y$). Then $x \pm y \notin A$, therefore $x \pm y \in B$ or $x \pm y \in C$. If we add $x \pm y$ into Y, we have a triad $(X, Y, Z)_1$, if we add $x \pm y$ into Z we have a triad $(X, Y, Z)_2$. By repeating the foregoing method we get a tree

$$(X, Y, Z)_0 \begin{cases} (X, Y, Z)_1 \begin{cases} (X, Y, Z)_{11} < \\ (X, Y, Z)_{12} < \end{cases} \\ (X, Y, Z)_2 \begin{cases} (X, Y, Z)_{21} < \\ (X, Y, Z)_{22} < \end{cases} \end{cases} \tag{2}$$

If we can find u, v to x, y, e.g. in Z such that $x \pm y = u \pm v$ ($u + v$ if $u \neq v$ or $u - v$ if $u \geqq v + n$, $u \neq 2v$), then $x \pm y$ can be placed only in Y and we add it into this set without extending the index of a triad X, Y, Z.

If the tree (2) ends by leaves $(X, Y, Z)_{i_1 i_2 \ldots}$ for which there exists a number n' so that

$$n' = x \pm y = s \pm t = u \pm v \tag{3}$$

where $x, y \in X$, $s, t \in Y$, $u, v \in Z$ (and x, y, s, t, u, v hold for the corresponding inequalities) and $n' \leqq 14n + 7$, then the supposed decomposition of A, B, C cannot exist.

The assertion is proved when we can construct a finite set of triads $(X, Y, Z)_0$ so that there exists a triad $(X, Y, Z)_0$ to each decomposition A, B, C which does not fulfil the assertion so that $X \subset A$, $Y \subset B$, $Z \subset C$, and so that to $(X, Y, Z)_0$ there exists a tree ending by leaves to which $n' \leqq 14n + 7$ exists such that n' fulfils (3).

In a concrete record of a tree we sign every vertex in the form

$$\begin{array}{rl} (i_1 i_2 \ldots) & X;\ X' | n' \\ & Y;\ Y' \\ & Z;\ Z' \end{array}$$

where X', Y', Z' designate the sets of numbers constructed by addition or subtraction of numbers from X, Y, Z, and which we can exactly place. If the corresponding vertex is a leaf of a tree, then we add to it the number n' fulfilling (3), separated by a strong line.

We underline one number in X, Y, Z which is not placed exactly in the previous step and which is competent for a bifurcation of a tree.

For example.

Proof of 15. Every decomposition of the interval $[2, 35]$ into three strongly sum-free sets A, B, C for which $[2, 4] \subset A$, $[5, 7] \subset B$ and $8 \notin B$ includes one of two following triads

$$(\{2, 3, 4\}, \{5, 6, 7\}, \{8\})_0, \quad (\{2, 3, 4, 8\}, \{5, 6, 7\}, \emptyset)_0$$

In the first case the tree (2) has the verteces:

(0) 2, 3, 4 5, 6, 7 8	(1) 2, 3, 4, $\underline{11}$ 5, 6, 7 8; 13	(2) 2, 3, 4 5, 6, 7 8, $\underline{11}$	(11) 2, 3, 4, 11, $\underline{21}$ 5, 6, 7 8, 13

(12) 2, 3, 4, 11; 27, 28 | 32 5, 6, 7, $\underline{21}$; 23, 29 8, 13; 15, 14, 17	(21) 2, 3, 4, $\underline{19}$ 5, 6, 7 8, 11	(22) 2, 3, 4 5, 6, 7, $\underline{19}$ 8, 11

(111) 2, 3, 4, 11, 21 | 15 5, 6, 7, $\underline{18}$; 10 8, 13; 24, 23	(112) 2, 3, 4, 11, 21; 30 | 32 5, 6, 7; 10, 25, 9, 26, 27 8, 13, $\underline{18}$; 17, 19, 15

(211) 2, 3, 4, 19; 10, 20 | 23
5, 6, 7, 15; 29
8, 11; 22, 21, 9, 14

(212) 2, 3, 4, 19; 28 | 32
5, 6, 7; 23, 9
8, 11, 15; 17, 16

(221) 2, 3, 4, 24
5, 6, 7, 19
8, 11; 26

(222) 2, 3, 4; 13, 14, 23, 22 | 21
5, 6, 7, 19; 16, 17, 18
8, 11, 24; 10, 9, 12

(2211) 2, 3, 4, 24; 15 | 17
5, 6, 7, 19, 22; 20
8, 11, 26; 27, 28

(2212) 2, 3, 4, 24; 14, 13, 32 | 34
5, 6, 7, 19; 10, 27
8, 11, 26, 22; 12, 16, 17, 21

In the second case

(0) 2, 3, 4, 8
5, 6, 7
∅; 11, 12

(1) 2, 3, 4, 8, 23
5, 6, 7
11, 12

(2) 2, 3, 4, 8
5, 6, 7, 23
11, 12

(11) 2, 3, 4, 8, 23 | 26
5, 6, 7, 20
11, 12; 15

(12) 2, 3, 4, 8, 23 | 10
5, 6, 7; 31, 15
11, 12, 20; 26, 25, 21

(21) 2, 3, 4, 8, 18; 28, 19, 13 | 25
5, 6, 7, 23; 14, 27, 10, 20
11, 12; 16, 30, 22, 24, 9

(22) 2, 3, 4, 8; 29, 30, 17 | 10
5, 6, 7, 24; 15
11, 12, 18; 28, 13, 21

For all n' found it holds that $n' \leqq 35$. Then there cannot exist a decomposition of the interval [2, 35] into three strongly sum-free sets A, B, C, so that $\{2, 3, 4\} \subset A$, $\{5, 6, 7\} \subset B$ and $8 \notin B$.

The longest proof from all assertions 13.—19. has the assertion 13.

Proof of 13. From every decomposition A, B, C, of the interval [2, 35] for which $\{2, 3, 4\} \not\subset A$ we choose one of the following four triads $(X, Y, Z)_0$:

$$(\{2, 3\}, \{4\}, \emptyset)_0, \quad (\{2, 4\}, \{3\}, \emptyset)_0,$$
$$(\{2\}, \{3, 4\}, \emptyset)_0, \quad (\{2\}, \{3\}, \{4\})_0$$

To reduce the number of bifurcations we choose also some other numbers from the interval [2, 35], e.g. 8, 9, into the sets X, Y, Z. For $(X, Y, Z)_0$ we get the following 36 possibilities:

$(\{2, 3, 8, 9\}, \{4\}, \emptyset)_0$, $(\{2, 3, 8\}, \{4, 9\}, \emptyset)_0$, $(\{2, 3, 8\}, \{4\}, \{9\})_0$, $(\{2, 3, 9\}, \{4, 8\}, \emptyset)_0$, $(\{2, 3\}, \{4, 8, 9\}, \emptyset)_0$, $(\{2, 3\}, \{4, 8\}, \{9\})_0$, $(\{2, 3, 9\}, \{4\}, \{8\})_0$, $(\{2, 3\}, \{4, 9\}, \{8\})_0$, $(\{2, 3\}, \{4\}, \{8, 9\})_0$, $(\{2, 4, 8, 9\}, \{3\}, \emptyset)_0$, $(\{2, 4, 8\}, \{3, 9\}, \emptyset)_0$, $(\{2, 4, 8\}, \{3\}, \{9\})_0$, $(\{2, 4, 9\}, \{3, 8\}, \emptyset)_0$, $(\{2, 4\}, \{3, 8, 9\}, \emptyset)_0$, $(\{2, 4\}, \{3, 8\}, \{9\})_0$, $(\{2, 4, 9\}, \{3\}, \{8\})_0$, $(\{2, 4\}, \{3, 9\}, \{8\})_0$, $(\{2, 4\}, \{3\}, \{8,9\})_0$, $(\{2, 8, 9\}, \{3, 4\}, \emptyset)_0$, $(\{2, 8\}, \{3, 4, 9\}, \emptyset)_0$, $(\{2, 8\}, \{3, 4\}, \{9\})_0$, $(\{2, 9\}, \{3, 4, 8\}, \emptyset)_0$, $(\{2\}, \{3, 4, 8, 9\}, \emptyset)_0$, $(\{2\}, \{3, 4, 8\}, \{9\})_0$,

$(\{2, 9\}, \{3, 4\}, \{8\})_0$, $(\{2\}, \{3, 4, 9\}, \{8\})_0$, $(\{2\}, \{3, 4\}, \{8, 9\})_0$, $(\{2, 8, 9\}, \{3\}, \{4\})_0$, $(\{2, 8\}, \{3, 9\}, \{4\})_0$, $(\{2, 8\}, \{3\}, \{4, 9\})_0$, $(\{2, 9\}, \{3, 8\}, \{4\})_0$, $(\{2\}, \{3, 8, 9\}, \{4\})_0$, $(\{2\}, \{3, 8\}, \{4, 9\})_0$, $(\{2, 9\}, \{3\}, \{4, 8\})_0$, $(\{2\}, \{3, 9\}, \{4, 8\})_0$, $(\{2\}, \{3\}, \{4, 8, 9\})_0$.

The 36 trees constructed to separate $(X, Y, Z)_0$ are introduced in [1, Supplement No. 3]. They are ended by about 250 leaves. In every leaf there is $n' \leqq 35$. Then for every decomposition A, B, C, of the interval $[2, 35]$ into three strongly sum-free sets $\{2, 3, 4\} \subset A$ holds.

The longest proof from all assertions 1.—25. has the assertion 1.

Proof of 1. Let $n \geqq 3$.

$$[n, 2n-1] = \bigcup_{1 \leqq k \leqq \frac{n}{2}} \{n, n+k, 2n-k\}$$

If A, B, C is a decomposition of an interval $[n, 14n+7]$ for which $[n, 2n-1] \not\subset \not\subset A$, then among the 4 following triads $(X, Y, Z)_0$ there always exists a triad such that $X \subset A$, $Y \subset B$, $Z \subset C$ for some k;

$$(\{n\}, \{n+k\}, \{2n-k\})_0, \quad (\{n\}, \{n+k, 2n-k\}, \emptyset)_0,$$
$$(\{n, 2n-k\}, \{n+k\}, \emptyset)_0, \quad (\{n, n+k\}, \{2n-k\}, \emptyset)_0.$$

To reduce the number of bifurcations in trees we arbitrarily add numbers $2n$ and $4n$ to every triad $(X, Y, Z)_0$. Again we get 36 possibilities for $(X, Y, Z)_0$, and they are as follows:

$(\{n, 4n\}, \{n+k, 2n-k\}, \{2n\})_0$, $(\{n\}, \{n+k, 2n-k, 4n\}, \{2n\})_0$, $(\{n\}, \{n+k, 2n-k\}, \{2n, 4n\})_0$, $(\{n, 2n, 4n\}, \{n+k\}, \{2n-k\})_0$, $(\{n, 2n\}, \{n+k, 4n\}, \{2n-k\})_0$, $(\{n, 2n\}, \{n+k\}, \{2n-k, 4n\})_0$, $(\{n, 4n\}, \{n+k\}, \{2n-k, 2n\})_0$, $(\{n\}, \{n+k, 4n\}, \{2n-k, 2n\})_0$, $(\{n\}, \{n+k\}, \{2n-k, 2n, 4n\})_0$, $(\{n, n+k, 2n, 4n\}, \{2n-k\}, \emptyset)_0$, $(\{n, n+k, 2n\}, \{2n-k, 4n\}, \emptyset)_0$, $(\{n, n+k, 2n\}, \{2n-k\}, \{4n\})_0$, $(\{n, n+k, 4n\}, \{2n-k, 2n\}, \emptyset)_0$, $(\{n, n+k\}, \{2n-k, 2n, 4n\}, \emptyset)_0$, $(\{n, n+k\}, \{2n-k, 2n\}, \{4n\})_0$, $(\{n, n+k, 4n\}, \{2n-k\}, \{2n\})_0$, $(\{n, n+k\}, \{2n-k, 4n\}, \{2n\})_0$, $(\{n, n+k\}, \{2n-k\}, \{2n, 4n\})_0$, $(\{n, 2n-k, 2n, 4n\}, \{n+k\}, \emptyset)_0$, $(\{n, 2n-k, 2n\}, \{n+k, 4n\}, \emptyset)_0$, $(\{n, 2n-k, 4n\}, \{n+k\}, \{4n\})_0$, $(\{n, 2n-k, 4n\}, \{n+k, 2n\}, \emptyset)_0$, $(\{n, 2n-k\}, \{n+k, 2n, 4n\}, \emptyset)_0$, $(\{n, 2n-k\}, \{n+k, 2n\}, \{4n\})_0$, $(\{n, 2n-k, 4n\}, \{n+k\}, \{2n\})_0$, $(n, 2n-k\}, \{n+k, 4n\}, \{2n\})_0$, $(\{n, 2n-k\}, \{n+k\}, \{2n, 4n\})_0$, $(\{n, 2n, 4n\}, \{n+k, 2n-k\}, \emptyset)_0$, $(\{n, 2n\}, \{n+k, 2n-k, 4n\}, \emptyset)_0$, $(\{n, 4n\}, \{n+k, 2n\}, \{2n-k\})_0$, $(\{n\}, \{n+k, 2n, 4n\}, \{2n-k\})_0$, $(\{n\}, \{n+k, 2n\}, \{2n-k, 4n\})_0$, $(\{n, 2n\}, \{n+k, 2n-k\}, \{4n\})_0$, $(\{n, 4n\}, \{n+k, 2n-k, 2n\}, \emptyset)_0$, $(\{n\}, \{n+k, 2n-k, 2n, 4n\}, \emptyset)_0$, $(\{n\}, \{n+k, 2n-k, 2n\}, \{4n\})_0$.

The 36 trees constructed to separate $(X, Y, Z)_0$ are introduced in [1, Supplement No. 2]. They are ended by about 600 leaves. In every leaf it holds that $n' \leqq 14n + 7$ for the number n', which fulfils (3). Then for every decomposition A, B, C of an interval $[n, 14n + 7]$ into three strongly sum-free sets it holds that $[n, 2n - 1] \subset A$.

In a construction of trees we have to care about $x = an + bk$, $y = en + dk$ in a construction of $x + y$ so that we have $an + bk \neq en + dk$, and in $x - y$ we have $an + bk \neq 2(en + dk)$ for all k, $1 \leqq k \leqq n/2$. Especially, we always use $n + k \neq 2n - k$, i.e. $n \neq 2k$. If n is even we inspect the position of $(3/2)n$ in a decomposition A, B, C separately. Further, we always use $b, d = 0, \pm 1$. As an example we introduce a tree beginning with $(\{n\}, \{n + k, 2n - k, 4n\}, \{2n\})_0$.

$$\begin{array}{ll}(0) & n \\ & n + k,\ 2n - k,\ 4n \\ & 2n\end{array} \qquad \begin{array}{ll}(1) & n,\ \underline{2n + k} \\ & n + k,\ 2n - k,\ 4n \\ & 2n\end{array}$$

$$\begin{array}{ll}(2) & n \\ & n + k,\ 2n - k,\ 4n \\ & 2n,\ \underline{2n + k}\end{array} \qquad \begin{array}{ll}(11) & n,\ 2n + k,\ \underline{5n + k} \\ & n + k,\ 2n - k,\ 4n \\ & 2n\end{array}$$

$$\begin{array}{ll}(12) & n,\ 2n + k;\ 7n + k,\ 3n \mid 8n + k \\ & n + k,\ 2n - k,\ 4n;\ 3n + k,\ 10n + k,\ 4n + k \\ & 2n,\ \underline{5n + k};\ 5n,\ 6n + k\end{array}$$

$$\begin{array}{ll}(21) & n,\ \underline{4n + k};\ 3n \mid 3n + k \\ & n + k,\ 2n - k,\ 4n;\ 7n + k \\ & 2n,\ 2n + k;\ 5n + k\end{array} \qquad \begin{array}{ll}(22) & n \\ & n + k,\ 2n - k,\ 4n,\ \underline{4n + k} \\ & 2n,\ 2n + k\end{array}$$

$$\begin{array}{ll}(111) & n,\ 2n + k,\ 5n + k;\ 5n,\ 8n \mid 9n \\ & n + k,\ 2n - k,\ 4n,\ \underline{6n + k};\ 7n + k \\ & 2n;\ 3n,\ 10n + k,\ 6n\end{array}$$

$$\begin{array}{ll}(112) & n,\ 2n + k,\ 5n + k;\ 8n + k,\ 8n \\ & n + k,\ 2n - k,\ 4n;\ 4n + k,\ 9n + k \\ & 2n,\ \underline{6n + k};\ 3n,\ 6n\end{array}$$

$$\begin{array}{ll}(221) & n,\ 6n \\ & n + k,\ 2n - k,\ 4n,\ 4n + k \\ & 2n,\ 2n + k\end{array} \qquad \begin{array}{ll}(222) & n;\ 8n + k \\ & n + k,\ 2n - k,\ 4n,\ 4n + k \\ & 2n,\ 2n + k,\ \underline{6n}\end{array}$$

$$\begin{array}{ll}(1121) & n,\ 2n + k,\ 5n + k,\ 8n + k,\ 8n \mid 9n \\ & n + k,\ 2n - k,\ 4n,\ 4n + k,\ 9n + k,\ \underline{7n + k} \\ & 2n,\ 6n + k,\ 3n,\ 6n\end{array}$$

$$\begin{array}{ll}(1122) & n,\ 2n + k,\ 5n + k,\ 8n + k,\ 8n \mid 13n + k \\ & n + k,\ 2n - k,\ 4n,\ 4n + k,\ 9n + k \\ & 2n,\ 6n + k,\ 3n,\ 6n,\ \underline{7n + k}\end{array}$$

$$\begin{array}{ll}(2211) & n,\ 6n,\ \underline{5n+k} \\ & n+k,\ 2n-k,\ 4n,\ 4n+k \\ & 2n,\ 2n+k\end{array} \qquad \begin{array}{ll}(2212) & n,\ 6n \\ & n+k,\ 2n-k,\ 4n,\ 4n+k \\ & 2n,\ 2n+k,\ \underline{5n+k}\end{array}$$

$$\begin{array}{ll}(2221) & n,\ 8n+k \\ & n+k,\ 2n-k,\ 4n,\ 4n+k,\ \underline{9n+k} \\ & 2n,\ 2n+k,\ 6n\end{array}$$

$$\begin{array}{ll}(2222) & n,\ 8n+k,\ 11n+k,\ 9n \mid 12n+k \\ & n+k,\ 2n-k,\ 4n,\ 4n+k;\ 7n+k,\ 8n \\ & 2n,\ 2n+k,\ 6n,\ \underline{9n+k};\ 3n\end{array}$$

$$\begin{array}{ll}(22111) & n,\ 6n,\ 5n+k,\ \underline{8n+k} \mid 11n+k \\ & n+k,\ 2n-k,\ 4n,\ 4n+k;\ 5n,\ 7n \\ & 2n,\ 2n+k;\ 3n,\ 9n+k\end{array}$$

$$\begin{array}{ll}(22112) & n,\ 6n,\ 5n+k;\ 10n+k,\ 3n \mid 13n+k \\ & n+k,\ 2n-k,\ 4n,\ 4n+k;\ 6n+k,\ 7n \\ & 2n,\ 2n+k,\ 8n+k;\ 5n,\ 11n+k\end{array}$$

$$\begin{array}{ll}(22121) & n,\ 6n,\ \underline{7n+k};\ 3n \mid 10n+k \\ & n+k,\ 2n-k,\ 4n,\ 4n+k;\ 6n+k \\ & 2n,\ 2n+k,\ 5n+k;\ 8n+k\end{array}$$

$$\begin{array}{ll}(22122) & n,\ 6n;\ 3n,\ 3n+k \mid 6n+k \\ & n+k,\ 2n-k,\ 4n,\ 4n+k,\ \underline{7n+k};\ 7n,\ 5n \\ & 2n,\ 2n+k,\ 5n+k;\ 9n,\ 3n-k\end{array}$$

$$\begin{array}{ll}(22211) & n,\ 8n+k,\ \underline{5n};\ 8n \mid 9n \\ & n+k,\ 2n-k,\ 4n,\ 4n+k,\ 9n+k;\ 7n+k \\ & 2n,\ 2n+k,\ 6n,\ 13n+k,\ 3n\end{array}$$

$$\begin{array}{ll}(22212) & n,\ 8n+k;\ 11n,\ 3n \mid 8n \\ & n+k,\ 2n-k,\ 4n,\ 4n+k,\ 9n+k \\ & 2n,\ 2n+k,\ 6n,\ \underline{5n}\end{array}$$

For $n = 1$ as by a decomposition into two strongly sum-free sets, also by a decomposition into three strongly sum-free sets the interval $[1, 14.1 + 7]$ is not the longest one, but the interval $[1, 23]$ is the longest one. One of decompositions is for example

$$A_0 = \{1, 2, 4, 8, 11, 22\}$$
$$B_0 = \{3, 5, 6, 7, 19, 21, 23\}$$
$$C_0 = \{9, 10, 12, 13, 14, 15, 16, 17, 18, 20\}.$$

All other decompositions differ from A_0, B_0, C_0 only slightly, the account of which is given in the following Theorem 2.

Theorem 2. Let A, B, C be a decomposition of the interval [1, 23] into three strongly sum-free sets and let $1 \in A$ and let B include the first number not belonging into A. Then

$$A \supset \{1, 2, 4, 8, 11, 22\}$$
$$B \supset \{3, 5, 6, 7, 19, 21, 23\}$$
$$C \supset \{9, 10, 12, 13, 14, 15, 18, 20\}$$

and

$$16, 17 \in C \quad \text{or} \quad 16 \in A, 17 \in C \quad \text{or} \quad 17 \in A, 16 \in C.$$

From Theorem 2 it immediately follows that the interval [1,23] is the longest one, as always 2, $22 \in A$, 3, $21 \in B$, 10, $14 \in C$ holds and

$$24 = 2 + 22 = 3 + 21 = 10 + 14.$$

Proof of Theorem 2. The complete proof is included in [1, Chapter 1]*. We present here only its separate steps. Let A, B, C be an arbitrary decomposition of the interval [1,23] into three strongly sum-free sets. Then

1. $\{1, 2\} \subset A$

Consequence:

2. $3 \in B$
3. If $\{1, 2\} \subset A$, $3 \in B$ then $4 \in A$
4. If $\{1, 2, 4\} \subset A$, $3 \in B$ then $5 \in B$
5. If $\{1, 2, 4\} \subset A$, $\{3, 5\} \subset B$ then $6 \in B$
6. If $\{1, 2, 4\} \subset A$, $\{3, 5, 6\} \subset B$ then $7 \in B$
7. If $\{1, 2, 4\} \subset A$, $\{3, 5, 6, 7\} \subset B$ then $8 \in A$.

Consequence:

8. $\{9, 10, 12\} \subset C$
9. If $\{1, 2, 4, 8\} \subset A$, $\{3, 5, 6, 7\} \subset B$, $\{9, 10, 12\} \subset C$ then $11 \in A$.

Consequence:

10. $13 \in C$
11. If $\{1, 2, 4, 8, 11\} \subset A$, $\{3, 5, 6, 7\} \subset B$, $\{9, 10, 12, 13\} \subset C$ then $14 \in C$.
12. If $\{1, 2, 4, 8, 11\} \subset A$, $\{3, 5, 6, 7\} \subset B$, $\{9, 10, 12, 13, 14\} \subset C$ then $15 \in C$.

So numbers from the interval [1, 15] are unambiguously decomposed into the sets A, B, C in the form

13. $A \supset \{1, 2, 4, 8, 11\}$
 $B \supset \{3, 5, 6, 7\}$
 $C \supset \{9, 10, 12, 13, 14, 15\}$

* see also [2].

14. If 13. then $16 \notin B$, $17 \notin B$.
15. If 13. then numbers 16, 17 cannot belong to A simultaneously.
16. If 13. then

$A \supset \{1, 2, 4, 8, 11, 22\}$
$B \supset \{3, 5, 6, 7, 19, 21, 23\}$
$C \supset \{9, 10, 12, 13, 14, 15, 18, 20\}$

Consequence:
17. $16, 17 \in C$ or $16 \in A$, $17 \in C$ or $17 \in A$, $16 \in C$.

The assertion 1. has again the longest proof from all the assertions 1.—16., and it is of the following structure:

If A, B, C is a decomposition of the interval [1, 23] for which $\{1, 2\} \not\subset A$, then $X \subset A$, $Y \subset B$, $Z \subset C$ for a triad $(X, Y, Z)_0 = (\{1\}, \{2\}, \emptyset)_0$. To reduce the number of bifurcations we add the triad of numbers 3, 4, 6 to the triad $(X, Y, Z)_0$ using all possible ways. We get 27 possibilities of $(X, Y, Z)_0$. The 27 trees constructed from them are introduced in [1, Supplement No. 1]. They contain about 150 leaves and each of them ends with $n' \leqq 23$. Therefore, $\{1, 2\} \subset A$ for every decomposition A, B, C of the interval [1, 23] into three strongly sum-free sets.

At the end we present several conjectures.

Let $[n, f(n, k)]$ be longest interval of integers decomposable into k strongly sum-free sets. Using the above $f(1, 2) = 8$, $f(n, 2) = 5n + 2$, $f(1, 3) = 23$, $f(n, 3) = 14n + 7$.

Conjecture 1. $f(1, 4) = 66$.

One of the decompositions of the interval [1, 66] into four strongly sum-free sets A_0, B_0, C_0, D_0 is

$$A_0 = \{1, 2, 4, 8, 11, 25, 50, 63\}$$
$$B_0 = \{3, 5, 6, 7, 19, 21, 23, 51, 52, 53, 64, 65, 66\}$$
$$C_0 = \{9, 10\} \cup [12, 18] \cup \{20\} \cup [54, 62]$$
$$D_0 = \{24\} \cup [26, 49].$$

Conjecture 2. $f(n, 4) = 41n + 21$ for $n \geqq 2$.

One of the decompositions of the interval $[n, 41n + 21]$ into four strongly sum-free sets is

$$\begin{aligned} A_0 = {} & [n, 2n] \cup [4n + 3, 5n + 2] \cup [10n + 7, 11n + 6] \cup \\ & \cup [13n + 8, 14n + 7] \cup [28n + 17, 29n + 16] \cup \\ & \cup [31n + 18, 32n + 17] \cup [37n + 21, 38n + 20] \cup \\ & \cup [40n + 22, 41n + 21] \end{aligned}$$
$$\begin{aligned} B_0 = {} & [2n + 1, 4n + 2] \cup [11n + 7, 13n + 7] \cup \\ & \cup [29n + 1, 31n + 17] \cup [38n + 21, 40n + 21] \end{aligned}$$

$$C_0 = [5n + 3, 10n + 6] \cup [32n + 18, 37n + 20]$$
$$D_0 = [14n + 8, 28n + 16]$$

Conjecture 3.

$$f(n, k) = k - 1 + (2^{k-1} + 1)n + \sum_{s=1}^{k-1} 2^{k-(s+1)}(f(n, s) + 1)$$

while

$$f(n, 1) = 2n.$$

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SÚHRN

ROZKLAD CELOČÍSELNÉHO INTERVALU NA TRI OSTRO SUMOVO-RIEDKE MNOŽINY

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V tejto práci sú zhrnuté výsledky z autorovej kandidátskej dizertačnej práce [1], podľa ktorej najdlhší celočíselný interval

$$[n, N] = \{n, n + 1, \dots, N\}$$

ktorý sa dá rozložiť na tri množiny tak, že ani v jednej nie je riešiteľná rovnica $x + y = z$, $x \neq y$, pre $n \geqq 2$ sa rovná $[n, 14n + 7]$ a pre $n = 1$ sa rovná $[1, 23]$.

РЕЗЮМЕ

РАЗБИЕНИЕ ЦЕЛОЧИСЛЕННОГО ПРОМЕЖУТКА НА ТРИ МНОЖЕСТВА, НЕ СОДЕРЖАЩИЕ СУММЫ СВОИХ ЧЛЕНОВ

Юлиус Бачик, Нитра

В работе излагаются результаты кандидатской диссертации автора [1], в которой доказано, что целочисленный промежуток

$$[n, N] = \{n, n + 1, \dots, N\}$$

максимальной длины, который разлагается на три такие множества, что ни в одном из них уравнение $x + y = z$, $x \neq y$ не имеет решения для $n \geqq 2$, равен $[n, 14n + 7]$ и для $n = 1$ равен $[1, 23]$.