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# EXTENSIONS OF SOME RESULTS ON DISJOINT COVERING SYSTEMS

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**Abstract.** We shall study DCS with couples of equal moduli. For a (2,2)-DCS (see the definition below), the moduli are determined by knowing which couples of moduli are equal. For a (2,2,2)-DCS we have five different forms of moduli. A conjecture shall be made about the least number of moduli, and some modulus of the form $2^a 3^b p^c$, where $a$, $b$ and $c$ are nonnegative intergers, and $p$ is a prime number greater than 3.

## I. Introduction

A system of congruences

$$a_j (\text{mod } n_j) \quad 0 \leqq a_j < n_j \qquad j = 1, 2, \ldots, k \tag{1}$$

where $k \geqq 2$ and $n_1 \leqq n_2 \leqq \ldots \leqq n_k$ is said to be a disjoint covering system (DCS) if every interger belongs to exactly one congruence in (1).

**Lemma 1.** If (1) is a DCS, then

a) the equality

$$\sum_{j=1}^{k} \frac{1}{n_j} = 1$$

holds;

b) for every $i, j = 1, 2, \ldots, k$, we have $(n_j, n_i) > 1$, where $(x, y)$ denotes the greater common divisor of numbers $x$ and $y$.

**Lemma 2.** If (1) is DCS and $z$ is a complex number, with $|z| < 1$. Then the equality

$$\frac{1}{1-z} = \frac{z^{a_1}}{1-z^{n_1}} + \frac{z^{a_2}}{1-z^{n_2}} + \ldots + \frac{z^{a_k}}{1-z^{n_k}}$$

holds.

**Lemma 3.** (Porubský) (see [2]). Let $p$, $b_1$, $b_2$, ..., $b_p$, $m$ be intergers with $2 \leqq p \leqq 5, 0 \leqq b_1 < b_2 < \ldots < b_p < m$. Let

$$\exp\left(\frac{2\pi i}{m} b_1\right) + \exp\left(\frac{2\pi i}{m} b_2\right) + \ldots + \exp\left(\frac{2\pi i}{m} b_p\right) = 0.$$

Let no partial sum of this sum vanish. Then

a) the congruence $b_1 \pmod{m/p}$ contains exactly those integers which belong to the system

$$b_1 \pmod{m},\ b_2 \pmod{m},\ \ldots,\ b_p \pmod{m}$$

if $p = 2$, 3 and 5;

b) the case $p = 4$ is impossible.

**Theorem 1.** (Davenport, Mirsky, Newman and Rado). If (1) is a DCS, then

$$n_{k-1} = n_k.$$

**Definition.** We say that a DCS is an $(m_1, m_2, \ldots, m_t)$-DCS if it has an $m_i$-tuple of equal moduli for each $i = 1, 2, \ldots, t$, with the remaining moduli being distinct, such that the $m_i$-equal moduli are smaller than the $m_{i+1}$-equal moduli, for $i = 1, 2, \ldots, t - 1$. For instance a (2,3)-DCS has a couple and a 3-tuple of equal moduli, such that

$$n_1 < n_2 < \ldots < n_{j-1} = n_j < n_{j+1} < \ldots < n_{k-2} = n_{k-1} = n_k$$

with $j < k - 2$.

**Theorem 2.** (Stein) (see [3]). If (1) is a (2)-DCS, then

$$n_i = 2^i \quad \text{for} \quad i = 1, 2, \ldots, k - 1, \quad n_k = 2^{k-1}.$$

**Theorem 3.** (Znám) (see [4]). If (1) is a (3)-DCS, then

$$n_i = 2^i \quad \text{for} \quad i = 1, 2, \ldots, k - 3, \quad n_{k-2} = n_{k-1} = n_k = 3 \cdot 2^{k-3}.$$

**Theorem 4.** (Porubský) (see [2]). If (1) is a (4)-DCS, then

$$n_i = 2^i \quad \text{for} \quad i = 1, 2, \ldots, k - 4, \quad n_{k-3} = n_{k-2} = n_{k-1} = n_k = 2^{k-2}$$

or

$$n_i = 2^i \quad \text{for} \quad i = 1, 2, \ldots, k - 5,\ n_{k-4} = 3 \cdot 2^{k-5},$$
$$n_{k-3} = n_{k-2} = n_{k-1} = n_k = 3 \cdot 2^{k-4}.$$

The following conjecture was made by Znám: in a (2,2)-DCS the moduli are of the form

$$2^a \cdot 3^b, \quad 0 \leqq b \leqq 1, \quad a \quad \text{are nonnegative integers.}$$

Further, Znám made the following question: in a (2,2,2)-DCS are the moduli of the form

$$2^a \cdot 3^b, \quad 0 \leqq b \leqq 2, \quad a \quad \text{are nonnegative integers.}$$

The problem is solved in the following theorems.

## II. The theorems

**Theorem** 5. Let (1) be a (2,2)-DCS such that

$$n_1 < n_2 < \ldots < n_{k-(w+1)} = n_{k-w} < n_{k-(w-1)} < \ldots < n_{k-1} = n_k. \tag{2}$$

Then we have

$$\begin{aligned} &n_i = 2^i \quad \text{for} \quad i = 1, 2, \ldots, k-(w+2) \\ &n_{k-(w+1)} = n_{k-w} = 3 \cdot 2^{k-(w+2)} \\ &n_i = 3 \cdot 2^{i-2} \quad \text{for} \quad i = k-(w-1), \ldots, k-2 \\ &n_{k-1} = n_k = 3 \cdot 2^{k-3}. \end{aligned} \tag{3}$$

**Proof.** By induction on $k$. The least $k$ for a (2,2)-DCS is 4. Let (1) be

$$a_j (\mathrm{mod}\ n_j) \quad j = 1, 2, 3, 4 \quad \text{with} \quad n_1 = n_2 < n_3 = n_4. \tag{4}$$

By Lemma 1, a) we have

$$\frac{1}{n_2} + \frac{1}{n_4} = \frac{1}{2}. \tag{5}$$

By (4) and (5) $n_2 < 4$. By (5) $n_2 > 2$. Therefore $n_2 = 3$ and $n_4 = 6$. The moduli of (4) fulfil (3) for $k = 4$ and $w = 2$. Assuming we have proved the theorem for $h < k$, we shall prove it for $k$, considering two different cases.

**A.** Let $w \geqq 3$. By Lemma 2 we have

$$\frac{1}{1-z} = \frac{z^{a_1}}{1-z^{n_1}} + \frac{z^{a_2}}{1-z^{n_2}} + \ldots + \frac{z^{a_{k-2}}}{1-z^{n_{k-2}}} + \frac{z^{a_{k-1}} + z^{a_k}}{1-z^{n_k}}. \tag{6}$$

Let $z$ tend to $\exp\left(\frac{2\pi i}{n_k}\right)$, with $|z| < 1$, in (6). Then, analogously to the proof of Theorem 1 in [4], we obtain

$$\exp\left(\frac{2\pi i}{n_k} a_{k-1}\right) + \exp\left(\frac{2\pi i}{n_k} a_k\right) = 0.$$

Suppose $a_{k-1} < a_k$, by Lemma 3 we obtain from (2) a DCS

$$\begin{aligned} &a_j (\text{mod } n_j) \quad j = 1, 2, \ldots, k-2 \\ &a_{k-1} \left(\text{mod } \frac{n_k}{2}\right) \end{aligned} \tag{7}$$

with $n_{k-(w+1)} = n_{k-w}$;
$w \geqq 3$ implies $n_{k-3} < n_{k-2}$ in (7), then by Theorem 1

$$\frac{n_k}{2} = n_{k-2}.$$

And then we have obtained a (2,2)-DCS with $k-1$ moduli. By induction hypothesis the moduli fulfil (3). Then making the change $n_k = 2n_{k-2}$, the moduli of (2) also fulfil (3).

**B.** Let $w = 2$. Then (2) becomes

$$n_1 < n_2 < \ldots < n_{k-4} < n_{k-3} = n_{k-2} < n_{k-1} = n_k. \tag{8}$$

i) Suppose $\frac{n_k}{2} < n_{k-2}$ and $\frac{n_k}{2} \neq n_j$ for any $j = 1, 2, \ldots, k-4$. Then (7) is a (2)-DCS. By Theorem 2 $\frac{n_k}{2} = 2^i$ for some $i = 1, 2, \ldots, k-3$, then $n_k = 2^{i+1} \leqq n_{k-2}$ which is a contradiction to (8).

ii) Suppose $\frac{n_k}{2} < n_{k-2}$ and $\frac{n_k}{2} = n_j$ for some $j = 1, 2, \ldots, k-4$. Then (7) is a (2,2)-DCS with $k-1$ moduli. By induction hypothesis $\frac{n_k}{2} = 3 \cdot 2^{j-1}$ for some $j = 1, 2, \ldots, k-4$; then $n_k = 3 \cdot 2^j$, $n_{k-2} = 3 \cdot 2^{k-4}$, which contradicts (8).

From i), ii) and the fact that $\frac{n_k}{2} > n_{k-2}$ contradicts Theorem 1, we have $\frac{n_k}{2} = n_{k-2}$. Then (7) is a (3)-DCS. By Theorem 3 the moduli are

$$n_i = 2^i \quad \text{for} \quad i = 1, 2, \ldots, k-4, \quad n_{k-3} = n_{k-2} = \frac{n_k}{2} = 3 \cdot 2^{k-4}.$$

Therefore the moduli of (2) fulfil (3) which proves the theorem.

**Example of a (2,2)-DCS**

0 (mod 2)
1 (mod 4)
3; 7 (mod 12)
11 (mod 24)
23; 47 (mod 48).

The moduli are of the form (3) for $k = 7$ and $w = 3$.

**Theorem 6.** Let (1) be a (2,3)-DCS such that

$$n_1 < n_2 < \dots < n_{k-(w+1)} = n_{k-w} < n_{k-(w-1)} < \dots < n_{k-2} = n_{k-1} = n_k. \quad (9)$$

Then we have one of the following cases:

$$\begin{aligned} &n_i = 2^i \quad \text{for} \quad i = 1, 2, \dots, k-(w+2) \\ &n_{k-(w+1)} = n_{k-w} = 3 \cdot 2^{k-(w+2)} \\ &n_i = 3 \cdot 2^{i-2} \quad \text{for} \quad i = k-(w-1), k-(w-2), \dots, k-3 \\ &n_{k-2} = n_{k-1} = n_k = 3^2 \cdot 2^{k-5}; \end{aligned} \quad (10)$$

$$\begin{aligned} &n_i = 2^i \quad \text{for} \quad i = 1, 2, \dots, k-6 \\ &n_{k-5} = 3 \cdot 2^{k-6} \\ &n_{k-4} = n_{k-3} = 3 \cdot 2^{k-5} \\ &n_{k-2} = n_{k-1} = n_k = 3^2 \cdot 2^{k-6}; \end{aligned} \quad (11)$$

$$\begin{aligned} &n_i = 2^i \quad \text{for} \quad i = 1, 2, \dots, k-5 \\ &n_{k-4} = n_{k-3} = 2^{k-3} \\ &n_{k-2} = n_{k-1} = n_k = 3 \cdot 2^{k-4}. \end{aligned} \quad (12)$$

**Proof.** We shall consider two different cases.

**A.** Let $w \geqq 4$. In an analogous way to the one we have used to prove $\frac{n_k}{2} = n_{k-2}$ for $w \geqq 3$ in Theorem 5 part **A**, we now can prove $\frac{n_k}{3} = n_{k-3}$.

Suppose $a_{k-2} < a_{k-1} < a_k$, then by Lemma 3 we obtain from (9) a DCS

$$\begin{aligned} &a_j \ (\text{mod } n_j) \quad j = 1, 2, \dots, k-3 \\ &a_{k-2} \left(\text{mod } \frac{n_k}{3}\right) \end{aligned} \quad (13)$$

which is a (2,2)-DCS with $k - 2$ moduli. By Theorem 5 the moduli of (13) are of the form (3). Therefore, after some changes the moduli of (9) fulfil (10).

**B.** Let $w = 3$. Then (9) becomes

$$n_1 < n_2 < \dots < n_{k-4} = n_{k-3} < n_{k-2} = n_{k-1} = n_k. \quad (14)$$

We shall study different cases:

i) If $\frac{n_k}{3} = n_{k-3}$, then we can modify (14) into a (3)-DCS. By Theorem 3 and changing $n_k$ for $3n_{k-3}$ we have obtain the moduli of (14) in the form of (10).

ii) If $n_{k-5} < \frac{n_k}{3} < n_{k-4}$, then we have a (2)-DCS. By Theorem 2 and after some changes we obtain the moduli of (14) in the form of (12).

iii) If $\frac{n_k}{3} = n_{k-5}$, then we can obtain a (2,2)-DCS with $k-2$ moduli, from (14). By Theorem 6 and after some changes we obtain the moduli of (14) in the form of (11).

iv) We now shall prove that i), ii), and iii) are the only possible cases. First, $\frac{n_k}{3} > n_{k-3}$ is impossible because of Theorem 1. Suppose $\frac{n_k}{3} < n_{k-5}$ and $\frac{n_k}{3}$ is different from all the smaller moduli, then we have a (2)-DCS. By Theorem 2 we have $n_{k-5} = 2^{k-4}$, $n_{k-4} = n_{k-3} = 2^{k-3}$ and $\frac{n_k}{3} = 2^t$ for some $t < k-4$. Then $n_k < 2^{t+2} \leqq 2^{k-3}$, which is a contradiction to (14). Now suppose $\frac{n_k}{3} < n_{k-2}$ and $\frac{n_k}{3} = n_j$ for some $j = 1, 2, \ldots, k-6$. Then we have a (2,2)-DCS with $k-2$ moduli, by Theorem 5

$$n_j = \frac{n_k}{3} = 3 \cdot 2^{j-1} \quad \text{for some} \quad j = 1, 2, \ldots, k-6 \quad \text{and}$$

$$n_{k-4} = n_{k-3} = 3 \cdot 2^{k-5}, \quad \text{then} \quad n_k < 3 \cdot 2^{j+1} \leqq 3 \cdot 2^{k-5} = n_{k-3}.$$

This is a contradiction to (14), which finishes the proof of our theorem.

**Examples of (2,3)-DCS.**

Form (10) for $k = 7$ and $w = 4$

0 (mod 2)
3; 5 (mod 6)
7 (mod 12)
1; 13; 25 (mod 36).

Form (11) for $k = 7$

0 (mod 2)
5 (mod 6)
1; 7 (mod 12)
3; 9; 15 (mod 18).

Form (12) for $k = 6$

$$\begin{gathered} 0 \pmod 2 \\ 3;\ 7 \pmod 8 \\ 1;\ 5;\ 9 \pmod{12}. \end{gathered}$$

**Theorem 7.** Let (1) be a (3,2)-DCS such that

$$n_1 < n_2 < \ldots < n_{k-(w+2)} = n_{k-(w+1)} = n_{k-w} < n_{k-(w-1)} < \ldots < n_{k-1} = n_k. \tag{15}$$

Then we have one of the following cases:

$$\begin{aligned} & n_i = 2^i \quad \text{for} \quad i = 1, 2, \ldots, k-(w+3) \\ & n_{k-(w+2)} = n_{k-(w+1)} = n_{k-w} = 2^{k-(w+1)} \\ & n_i = 2^{i-1} \quad \text{for} \quad i = k-(w-1), k-(w-2), \ldots, k-2 \\ & n_{k-1} = n_k = 2^{k-2}; \end{aligned} \tag{16}$$

$$\begin{aligned} & n_i = 2^i \quad \text{for} \quad i = 1, 2, \ldots, k-(w+4) \\ & n_{k-(w+3)} = 3 \cdot 2^{k-(w+4)} \\ & n_{k-(w+2)} = n_{k-(w+1)} = n_{k-w} = 3 \cdot 2^{k-(w+3)} \\ & n_i = 3 \cdot 2^{i-3} \quad \text{for} \quad i = k-(w-1), k-(w-2), \ldots, k-2 \\ & n_{k-1} = n_k = 3 \cdot 2^{k-4}; \end{aligned} \tag{17}$$

$$\begin{aligned} & n_i = 2^i \quad \text{for} \quad i = 1, 2, \ldots, k-(w+5) \\ & n_{k-(w+4)} = 3 \cdot 2^{k-(w+5)};\ n_{k-(w+3)} = 3 \cdot 2^{k-(w+4)} \\ & n_{k-(w+2)} = n_{k-(w+1)} = n_{k-w} = 3^2 \cdot 2^{k-(w+5)} \\ & n_i = 3 \cdot 2^{i-4} \quad \text{for} \quad i = k-(w-1), k-(w-2), \ldots, k-2 \\ & n_{k-1} = n_k = 3 \cdot 2^{k-5}; \end{aligned} \tag{18}$$

$$\begin{aligned} & n_i = 2^i \quad \text{for} \quad i = 1, 2, \ldots, k-(w+4) \\ & n_{k-(w+3)} = 2^{k-(w+2)} \\ & n_{k-(w+2)} = n_{k-(w+1)} = n_{k-w} = 3 \cdot 2^{k-(w+3)} \\ & n_i = 2^{i-2} \quad \text{for} \quad i = k-(w-1), k-(w-2), \ldots, k-2 \\ & n_{k-1} = n_k = 2^{k-3}. \end{aligned} \tag{19}$$

**Proof.** By induction on $k$. The least $k$ for a (3,2)-DCS is 5. Let (1) be

$$a_j \pmod{n_j} \quad j = 1, 2, 3, 4, 5 \quad \text{with} \quad n_1 = n_2 = n_3 < n_4 = n_5. \tag{20}$$

By Lemma 1, a). We have

$$\frac{3}{n_3} + \frac{2}{n_5} = 1. \tag{21}$$

By (20) and (21) we have $3 < n_3 < n_5$ and $n_3 < 5$. Therefore $n_3 = 4$ and $n_5 = 8$. Then the moduli of (20) fulfil (16) for $k = 5$ and $w = 2$. Assuming we have proved the theorem for $h < k$ we shall prove it for $k$. As above, we consider two different cases.

**A.** Let $w \geqq 3$ then, analogously to **A** of Theorem 5 we obtain $\frac{n_k}{2} = n_{k-2}$. Then we have a (3,2)-DCS with $k - 1$ moduli. By induction hypothesis the moduli are of one of the forms (16) through (19). Changing $n_k$ for $2n_{k-2}$ we find that the moduli of (15) are of the required form.

**B.** Let $w = 2$. Then (15) becomes

$$n_1 < n_2 < \ldots < n_{k-4} = n_{k-3} = n_{k-2} < n_{k-1} = n_k. \qquad (22)$$

We shall study different cases:

i) If $\frac{n_k}{2} = n_{k-2}$, then we can obtain a (4)-DCS. By Theorem 4 and after some changes, the moduli of (22) are of the form (16) or (17).

ii) If $\frac{n_k}{2} = n_j$ for some $j = 1, 2, \ldots, k - 5$, then we can obtain a (2,3)-DCS, from (22). By Theorem 6 the moduli of (22) are of the form (10), (11) or (12).

a) If they are of the form (10), then we have: $\frac{n_k}{2} = 3 \cdot 2^{j-1}$ for some $j = 1, 2, \ldots, k - 5$ and $n_{k-2} = 3^2 \cdot 2^{k-6}$. Then $n_k = 3 \cdot 2^j < n_{k-2}$, which is a contradiction to (22).

b) If they are of the form (11), then we have: $\frac{n_k}{2} = 3 \cdot 2^{k-6}$ and $n_{k-2} = 3^2 \cdot 2^{k-7}$. Then the moduli of (22) are of the form (18) for $w = 2$.

c) If they are of the form (12) then we have: $\frac{n_k}{2} = n_{k-5} = 2^{k-4}$ and $n_{k-2} = 3 \cdot 2^{k-5}$. Then the moduli of (22) are of the form (19) for $w = 2$.

iii) If $\frac{n_k}{2} < n_{k-2}$ and $\frac{n_k}{2} \neq n_j$ for any $j = 1, 2, \ldots, k - 5$ we have a (3)-DCS. By Theorem 3, $\frac{n_k}{2} = 2^i$ for some $i = 1, 2, \ldots, k - 4$, and $n_{k-2} = 3 \cdot 2^{k-4}$. Then $n_k < n_{k-2}$, which contradicts (22).

With i), ii), iii) and the fact that $\frac{n_k}{2} > n_{k-2}$ contradicts Theorem 1, we have finished the proof of the theorem.

**Examples of a (3,2)-DCS.**

Form (16) for $k = 6$ and $w = 2$

$$\begin{aligned} &0 \pmod 2\\ 1;\ 3;\ &5 \pmod 8\\ 7;\ &15 \pmod{16}. \end{aligned}$$

Form (17) for $k = 6$ and $w = 2$

$$\begin{aligned} &0 \pmod 3\\ 1;\ 2;\ &5 \pmod 6\\ 4;\ &10 \pmod{12}. \end{aligned}$$

Form (18) for $k = 8$ and $w = 2$

$$\begin{aligned} &0 \pmod 2\\ &1 \pmod 6\\ &5 \pmod{12}\\ 3;\ 9;\ &15 \pmod{18}\\ 11;\ &23 \pmod{24}. \end{aligned}$$

Form (19) for $k = 6$ and $w = 2$

$$\begin{aligned} &1 \pmod 4\\ 0;\ 2;\ &4 \pmod 6\\ 3;\ &7 \pmod 8). \end{aligned}$$

**Theorem 8.** Let (1) be a (2,2,2)-DCS, such that

$$\begin{aligned} &n_1 < n_2 < \dots < n_{k-w-(z+1)} = n_{k-w-z} < n_{k-w-(z-1)} < \dots\\ &\dots < n_{k-(w+1)} = n_{k-w} < n_{k-(w-1)} < \dots < n_{k-1} = n_k. \end{aligned} \tag{23}$$

Then we have one of the following cases:

$$\begin{aligned} &n_i = 2^i \quad \text{for} \quad i = 1, 2, \dots, k - w - (z+2)\\ &n_{k-w-(z+1)} = n_{k-w-z} = 3 \cdot 2^{k-w-(z+2)}\\ &n_i = 3 \cdot 2^{i-2} \quad \text{for} \quad i = k - w - (z-1),\ k - w - (z-2),\ \dots,\ k - (w+2)\\ &n_{k-(w+1)} = n_{k-w} = 3^2 \cdot 2^{k-(w+4)}\\ &n_i = 3^2 \cdot 2^{i-4} \quad \text{for} \quad i = k - (w-1),\ k - (w-2),\ \dots,\ k-2\\ &n_{k-1} = n_k = 3^2 \cdot 2^{k-5}; \end{aligned} \tag{24}$$

$$\begin{aligned} &z = 2 \quad \text{and}\\ &n_i = 2^i \quad \text{for} \quad i = 1, 2, \dots, k - (w+5)\\ &n_{k-(w+4)} = 3 \cdot 2^{k-(w+5)} \end{aligned}$$

$$n_{k-(w+3)} = n_{k-(w+2)} = 3\cdot 2^{k-(w+4)}$$
$$n_{k-(w+1)} = n_{k-w} = 3^2\cdot 2^{k-(w+5)} \tag{25}$$
$$n_i = 3^2\cdot 2^{i-5} \quad \text{for} \quad i = k-(w-1),\ k-(w-2),\ \dots,\ k-2$$
$$n_{k-1} = n_k = 3^2\cdot 2^{k-6};$$

$z = 2$ and
$$n_i = 2^i \quad \text{for} \quad i = 1, 2, \dots, k-(w+4)$$
$$n_{k-(w+3)} = n_{k-(w+2)} = 2^{k-(w+2)}$$
$$n_{k-(w+1)} = n_{k-w} = 3\cdot 2^{k-(w+3)} \tag{26}$$
$$n_i = 3\cdot 2^{i-3} \quad \text{for} \quad i = k-(w-1),\ k-(w-2),\ \dots,\ k-2$$
$$n_{k-1} = n_k = 3\cdot 2^{k-4};$$

$z = 2$ and
$$n_i = 2^i \quad \text{for} \quad i = 1, 2, \dots, k-(w+6)$$
$$n_{k-(w+5)} = 3\cdot 2^{k-(w+6)}$$
$$n_{k-(w+4)} = 3\cdot 2^{k-(w+5)}$$
$$n_{k-(w+3)} = n_{k-(w+2)} = 3^2\cdot 2^{k-(w+6)} \tag{27}$$
$$n_{k-(w+1)} = n_{k-w} = 3\cdot 2^{k-(w+4)}$$
$$n_i = 3^2\cdot 2^{i-6} \quad \text{for} \quad i = k-(w-1),\ k-(w-2),\ \dots,\ k-2$$
$$n_{k-1} = n_k = 3^2\cdot 2^{k-7};$$

$z = 2$ and
$$n_i = 2^i \quad \text{for} \quad i = 1, 2, \dots, k-(w+5)$$
$$n_{k-(w+4)} = 2^{k-(w+3)}$$
$$n_{k-(w+3)} = n_{k-(w+2)} = 3\cdot 2^{k-(w+4)} \tag{28}$$
$$n_{k-(w+1)} = n_{k-w} = 2^{k-(w+2)}$$
$$n_i = 3\cdot 2^{i-4} \quad \text{for} \quad i = k-(w-1),\ k-(w-2),\ \dots,\ k-2$$
$$n_{k-1} = n_k = 3\cdot 2^{k-5}.$$

**Proof.** By induction on $k$. The least $k$ for a (2,2,2)-DCS is 6. Let (23) be

$$a_j \ (\text{mod } n_j) \quad j = 1, 2, \dots, 6 \qquad \text{with} \qquad n_1 = n_2 < n_3 = n_4 < n_5 = n_6. \tag{29}$$

By Lemma 1, a) we have

$$\frac{1}{n_2} + \frac{1}{n_4} + \frac{1}{n_6} = \frac{1}{2}. \tag{30}$$

By (29) and (30) we have

$$2 < n_2 < n_4 < n_6. \tag{31}$$

By (30), (31) and Lemma 1, b) we have $2 < n_2 < 5$.

a) Let $n_2 = 3$, then we have $\frac{1}{n_4} + \frac{1}{n_6} = \frac{1}{6}$ by (31), then $6 < n_4 < n_6$. Therefore $n_4 < 12$. By Lemma 1, b) we have $(n_2, n_4) > 1$, then $n_4 = 9$, and then $n_6 = 18$. So the moduli of (29) fulfil (24), for $k = 6$, $w = 2$ and $z = 2$.

b) Let $n_2 = 4$, then we have $\frac{1}{n_4} + \frac{1}{n_6} = \frac{1}{4}$. So $4 < n_4 < n_6$ and then $n_4 < 8$. By Lemma 1, b) we have $(n_2, n_4) > 1$. Then $n_4 = 6$ and $n_6 = 12$. So the moduli of (29) fulfil (26) for $k = 6$ and $w = 2$.

Assuming we have proved the theorem for $h < k$ we shall prove it for $k$. As above, we consider two different cases.

**A.** Let $w \geqq 3$. Analogously to **A** of Theorem 5 we can prove that $\frac{n_k}{2} = n_{k-2}$. Then we have a (2,2,2)-DCS with $k - 1$ moduli. By induction hypothesis the moduli are of the required form. Changing $n_k$ for $2n_{k-2}$ we have that the moduli of (23) are of one of the forms (24) through (28).

**B.** Let $w = 2$. Then (23) becomes

$$\begin{aligned} n_1 < n_2 < \ldots < n_{k-2-(z+1)} = n_{k-2-z} < n_{k-2-(z-1)} < \ldots \\ \ldots < n_{k-3} = n_{k-2} < n_{k-1} = n_k. \end{aligned} \tag{33}$$

We shall study different cases:

i) Let $\frac{n_k}{2} = n_{k-2}$, then we obtain (from (33)) a (2,3)-DCS

$$\begin{aligned} n_1 < n_2 < \ldots < n_{k-2-(z+1)} = n_{k-2-Z} < n_{k-2-(z-1)} < \ldots \\ \ldots < n_{k-3} = n_{k-2} = \frac{n_k}{2}. \end{aligned} \tag{34}$$

By Theorem 6 the moduli of (34) must be of one of the following forms:

a) Form (10), then we have

$$n_i = 2^i \quad \text{for } i = 1, 2, \ldots, k - 2 - (z + 2)$$
$$n_{k-2-(z+1)} = n_{k-2-z} = 3 \cdot 2^{k-2-(z+2)}$$
$$n_i = 3 \cdot 2^{i-2} \quad \text{for } i = k - 2 - (z - 1), k - 2 - (z - 2), \ldots, k - 4$$
$$\frac{n_k}{2} = n_{k-3} = n_{k-2} = 3^2 \cdot 2^{k-6}.$$

Then the moduli of (33) are of the form (24) for $w = 2$.

b) Form (11); then $z$ must be equal to 2 and

$$n_i = 2^i \quad \text{for } i = 1, 2, \ldots, k-7$$
$$n_{k-6} = 3 \cdot 2^{k-7}$$
$$n_{k-5} = n_{k-4} = 3 \cdot 2^{k-6}$$
$$\frac{n_k}{2} = n_{k-3} = n_{k-2} = 3^2 \cdot 2^{k-7}.$$

Hence the moduli of (33) are of the form (25) for $w = 2$.

c) Form (12). Then $z = 2$ and

$$n_i = 2^i \quad \text{for } i = 1, 2, \ldots, k-6$$
$$n_{k-5} = n_{k-4} = 2^{k-4}$$
$$\frac{n_k}{2} = n_{k-3} = n_{k-2} = 3 \cdot 2^{k-5}.$$

Hence the moduli of (33) are of the form (26) for $w = 2$.

ii) Let $\frac{n_k}{2} = n_{k-2-z}$, then we can obtain (from (33)) a (3,2)-DCS with $k-1$ moduli

$$n_1 < n_2 < \ldots < n_{k-2-(z+1)} = n_{k-2-z} = \frac{n_k}{2} < n_{k-2-(z-1)} < \ldots$$
$$\ldots < n_{k-3} = n_{k-2}. \tag{35}$$

By Theorem 7 the moduli of (35) must be of one of the following forms:

a) Form (16). Then we have

$$\frac{n_k}{2} = n_{k-2-z} = n_{k-2-(z+1)} = 2^{k-2-z} \text{ and}$$
$$n_{k-2} = 2^{k-3}. \text{ Then } n_k = 2^{k-(z+1)} > 2^{k-3} = n_{k-2}$$

implies $z < 2$. A contradiction to (33).

b) Form (17). Then we have

$$\frac{n_k}{2} = 3 \cdot 2^{k-2-(z+2)} \text{ and } n_{k-2} = 3 \cdot 2^{k-5}.$$
$$n_k = 3 \cdot 2^{k-(z+3)} > 3 \cdot 2^{k-5} = n_{k-2} \text{ implies } z < 2.$$

A contradiction to (33).

c) Form (18). The we have

$\frac{n_k}{2} = 3^2 \cdot 2^{k-2-(z+4)}$ and $n_{k-2} = 3 \cdot 2^{k-6}$.

$n_k = 3^2 \cdot 2^{k-(z+5)} > 3 \cdot 2^{k-6} = n_{k-2}$ implies $z \leqq 2$.

As $z = 1$ cannot be, we have $z = 2$.
Therefore (35) becomes

$$n_1 < n_2 < \ldots < n_{k-5} = n_{k-4} = \frac{n_k}{2} < n_{k-3} = n_{k-2} \tag{36}$$

with

$n_i = 2^i$ for $i = 1, 2, \ldots, k-8$

$n_{k-7} = 3 \cdot 2^{k-8}$

$n_{k-6} = 3 \cdot 2^{k-7}$

$n_{k-5} = n_{k-4} = \frac{n_k}{2} = 3^2 \cdot 2^{k-8}$

$n_{k-3} = n_{k-2} = 3 \cdot 2^{k-6}$.

Then the moduli of (33) are of the form (27) form (27) for $w = 2$.

d) Form (19). Then we have

$\frac{n_k}{2} = 3 \cdot 2^{k-2-(z+2)}$ and $n_{k-2} = 2^{k-4}$.

$n_k = 3 \cdot 2^{k-(z+3)} > 2^{k-4}$ implies $z \leqq 2$. Then $z = 2$

and $n_i = 2^i$ for $i = 1, 2, \ldots, k-7$

$n_{k-6} = 2^{k-5}$

$n_{k-5} = n_{k-4} = \frac{n_k}{2} = 3 \cdot 2^{k-6}$

$n_{k-3} = n_{k-2} = 2^{k-4}$.

Therefore the moduli of (33) are of the form (28) for $w = 2$.

iii) Let $\frac{n_k}{2} = n_j$ for some $j$ such that $n_j$ only occurs once in (33). Then we can obtain, from (33), a (2,2,2)-DCS with $k-1$ moduli. By induction hypothesis this DCS will have the moduli of one of the forms (24) through (28). Making the change $n_k$ for $2n_j$ we obtain the moduli of (33) in the required form.

iv) Let $\frac{n_k}{2} \neq n_j$ for any $j = 1, 2, \ldots, k-2$ in (33). Then we can obtain a (2,2)-DCS:

a) If $\frac{n_k}{2} < n_{k-2-z}$, then $n_{j-1} < \frac{n_k}{2} < n_j$ for some $j = 1, 2, \ldots, k - 2 - (z + 1)$ which implies, by Theorem 5, $\frac{n_k}{2} \leqq 2^{k-2-(z+1)}$ but $n_{k-2} = 3 \cdot 2^{k-4}$, which implies $z = 0$. A contradiction to (33).

b) If $n_{k-2-z} < \frac{n_k}{2} < n_{k-3}$, then $\frac{n_k}{2} = 3 \cdot 2^{j-2}$ for some $j = k - 2 - (z - 1)$, $k - 2 - (z - 2), \ldots, k - 3$. Then $n_k = 3 \cdot 2^{j-1} \leqq 3 \cdot 2^{k-4} = n_{k-2}$. A contradiction to (33). This finishes the proof of the theorem.

**Examples of (2,2,2)-DCS.**

Form (24) for $k = 9$, $w = 3$ and $z = 3$

0 (mod 2)
3; 5 (mod 6)
7 (mod 12)
13; 25 (mod 36)
37 (mod 72)
1; 73 (mod 144).

Form (25) for $k = 9$ and $w = 3$

0 (mod 2)
5 (mod 6)
3; 9 (mod 12)
7; 13 (mod 18)
19 (mod 36)
1; 37 (mod 72).

Form (26) for $k = 9$ and $w = 3$

0 (mod 2)
1; 5 (mod 8)
7; 11 (mod 12)
15 (mod 24)
3; 27 (mod 48).

Form (27) for $k = 10$ and $w = 3$

0 (mod 2)
5 (mod 6)
7 (mod 12)

$$9;\ 15 \pmod{18}$$
$$1;\ 13 \pmod{24}$$
$$21 \pmod{36}$$
$$3;\ 39 \pmod{72}.$$

Form (28) for $k = 9$ and $w = 3$

$$0 \pmod 2$$
$$1 \pmod 8$$
$$7;\ 11 \pmod{12}$$
$$5;\ 13 \pmod{16}$$
$$15 \pmod{24}$$
$$3;\ 27 \pmod{48}.$$

All the DCS in the theorems are Natural Covering Systems (see definition in [2′]).

**Remark.** In the paper [4] Znám made the following (false) conjecture: If in a DCS there exist only pairs of equal moduli (no three being equal) then all moduli are of the form $2^a \cdot 3^b$, where $a$ and $b$ are nonnegative integers. N. Burshtein and J. Schönheim (see [1]) found a counter-example to this conjecture constructing a DCS with 8 pairs or equal moduli (no three moduli being equal) and with the moduli of the form $2^a \cdot 3^b \cdot 5^c$, where $a$, $b$, and $c$ are nonnegative integers.

It seems possible that by the methods used in the proofs of the above theorems, it can be proved that a DCS with only couples of equal moduli, up to 7 couples, has the moduli of the form $2^a \cdot 3^b$, $a$ and $b$ being nonnegative integers. Then, 16 will be the least number of moduli (i.e., 8 couples of moduli) for a DCS satisfying:

1. There are only couples of equal moduli.
2. A prime number greater than 3 divides at least one modulus.

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## SÚHRN

## ZOBECNENIE NIEKTORÝCH VÝSLEDKOV O PRESNE POKRÝVAJÚCICH SÚSTAVÁCH

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V práci je okrem iného ukázané, že ak v presne pokrývajúcej sústave sú niektoré dve dvojice modulov rovnaké, potom všetky moduly majú tvar $2^a3^b$. Podobný výsledok je ukázaný aj pre presne pokrývajúce sústavy, v ktorých sú rovnaké jedna dvojica a jedna trojica modulov, ako aj pre tri dvojice rovnakých modulov.

## РЕЗЮМЕ

## ОБОБЩЕНИЕ НЕКОТОРЫХ РЕЗУЛЬТАТОВ КАСАЮЩИХСЯ ДИЗЪЮНКТНЫХ ПОКРЫВАЮЩИХ СИСТЕМ

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В работе кроме прочего доказана следующая теорема: если в дизъюнктной покрывающей системе две пары модулей совпадают, то все модули имеют вид $2^a3^b$. Аналогичный результат доказан и для дизъюнктных покрывающих систем, в которых совпадает одна пара и одна тройка модулей, или для трецх пар совпадающих модулей.