UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XLVIII—XLIX — 1986

REAL NUMBERS AT THE SECONDARY SCHOOL AND AT THE UNIVERSITY, I.

EVA NYULASSYOVÁ — OTO STRAUCH, Bratislava

Introduction

Constructing the real numbers we shall point out that quality of theirs which will help us express the quantitative relations between objects and world, i.e. the fact that they can be compared, added, multiplied and divided. To each physical quantity which can be measured a number is attributed. In formal opinion we can summarize these demands into axioms and we can construct this theory axiomatically. That allows us to find out which properties of the solid of real numbers are fundamental and thus we can penetrate easier into the structure of proofs. We construct here the real numbers from objects that were constructed beforehand. Concrete models of real numbers warrant out theory is not contradictory and also they show us the way how to table the real numbers to the physical quantity in the act of measurement.

In our view, the introduction of the theory of real numbers into the curriculum at secondary schools by using models constructed by the theory of sets (for example: with the help of Dedekind cuts or using classes of Cauchy-sequences) is not advisable. As for thinking it is too exact for very big abstraction which is necessary to the notion of a real number. Further, this abstraction hinders an exercise of operations with real numbers and algorithms necessary to the numerical calculus of special functions and their evaluations. Also the axiomatical method is not advisable for the curriculum of secondary schools because it misses continuities and it turns an attention out of concrete calculus.

In this paper we shall give an outline of the school programme of the theme "Real numbers" for secondary schools and for universities. As for the course of mathematics at secondary schools, the main idea is given in an identification of real numbers with decimal expansions with respect to the paper [1] (also see [7]). As for the course of mathematics in universities, the basic idea is located in an axiomatic method which is connected with a model constructed through Cauchy-sequences, i.e. by using the theorem of existence of a closure of a metric space. We prefer this model to others because it gives us the construction of p-addic numbers as well (see [2]). We shall deal not only with the proof of the unicity of the field of real numbers (i.e. of categoricment of axioms) but also with the proof of the unicity of the fields of complex numbers, quaternions and Cayley numbers. We shall start from the field of rational numbers whose properties we shall consider well-known.

Real numbers at secondary schools

At present in text-books [3], [4], the real numbers are presented as points on the numerical axis [3, p. 127]. It gives a possibility to construct the sum, product and quotient geometrically and to show the existence of a calculus of the second root of a number [4, p. 146]. This procedure of a construction with the help of a ruler and circular cannot be used by the higher roots of numbers, by the evaluation of exponential, logarithmical and trigonometrical functions. There are indroduced no algorithms for an evaluation of elementary functions in these text-books (without [4, p. 170—171]). The authors of [3], [4] stand on the positions of an actual infinity. From this aspect, the function is something finished, evaluated-tabled once forever. The function is the set of couples (in which no couple has the first co-ordinate the same and the second co-ordinate different simultaneously [4, p. 15]), which are selected according to some common property (for example: they fulfil some equality [4, p. 30]). The verification of this property was made for all couples long time ago; the way of verification is not important. From this point of view, in the text-books [3], [4] only the algebraic properties of elementary functions are exercised.

In this didactic remark we shall turn the reader's attention to the fact that the algorithm of developing real number into a decimal expansion gives us the possibility to construct the theory of real numbers as a theory of decimal expansions in which the elementary functions will be introduced with the help of that algorithm. We think that the idea of a function as a set of couples on the one hand and as an algorithm-calculation on the other hand are two equall--valued sides of the same matter and they should be substituted equally in secondary text-books.

As the real number α we shall understand an arbitrary formal infinite sum on the right side of an equality

$$\alpha = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \ldots + a_0 + \frac{a_{-1}}{10} + \frac{a_{-2}}{10^2} + \ldots$$

or its record in a shorter form:

$$\alpha = a_n a_{n-1} \ldots a_0,\ a_{-1},\ a_{-2} \ldots \tag{1}$$

where $a_n \neq 0$ as $n > 0$, $0 \leqq a_i \leqq 9$ for all $i = n, n-1, \ldots, 0, -1, -2, \ldots$. In the following we shall say that the right side of (1) is called a decimal form of a number α or a decimal expansion of a number α. The non-negative integers a_{-k} are called digits of the k-order in this expansion. The two following expansions

$$\begin{aligned} &a_n a_{n-1} \ldots a_0,\ a_{-1} a_{-2} \ldots a_{-k} 999 \ldots 9 \ldots \\ &a_n a_{n-1} \ldots a_0,\ a_{-1} a_{-2} \ldots (a_{-k} + 1) 000 \ldots 0 \ldots, \end{aligned} \tag{2}$$

where $a_{-k} < 9$ can be identified because they present one real number only. We will speak about a finite decimal expansion if there exist only zeros behind some digit in that decimal expansion; sometimes we can omit the zeros in a record (1).

If those decimal expansions would be real numbers they have to satisfy the following properties:

(a) For each physical quantity we can table numbers in a decimal representation by a measurement, especially for longitudes.

(b) All rational numbers can be written in a decimal form. There are decimal expansions which present no rational numbers.

(c) We can find out which one of two real numbers in a decimal representation is a greater than the other or when they are equivalent. We can insert the third number in a decimal form between two different real numbers.

(d) We can find a limit (if it exists) in a decimal form to the sequence of numbers in a decimal form. We can find the bound of a bounded set of real numbers in a decimal representation.

(e) We can find the sum, difference, product and the quotient in a decimal form for each two real numbers in a decimal form so that the commutative, asociative and distributive laws are fulfilled. We can find also the inverse element for addition and multiplication.

(f) We can calculate values of elementary functions for each real number in a decimal representation once more in a decimal representation.

In the following, we shall distribute the properties (a)—(f) in sections in that way as they will be introduced in secondary schools. We shall always accent the numerical-constructive side of investigated objects. We shall not consider suitable to introduce the total proofs of an existence of objects (limits, functions) in a course of mathematics in a secondary school. We shall introduce them in a part of an university-course of math which will be prepared. The proofs will be made on a basis of decimal expansions which will be present by separate classes of Cauchy-sequences.

(a) **The measurement of abscissas.** Let us choose a length of the abscissa AB as a unit length. We have an arbitrary abscissa CD and we denote its length α. We can measure the length of the abscissa CD with the help of abscissa AB and we can express a result of measurement in a decimal form in the following way:

We denote by $[\alpha]$ a number of possibilities to place the abscissa AB on CD and we denote by $\{\alpha\}$ the length of that abscissa in CD which rests after this placement (let it be C_1D).

Similarly, we denote by $[10\{\alpha\}]$ a number of possibilities to place the abscissa AB on an abscissa which is 10-times longer than C_1D and by $\{10\{\alpha\}\}$ we denote the rest of this placement, and so one. Then we have

$$\begin{aligned} \alpha &= [\alpha] + \{\alpha\} = [\alpha] + \frac{10\{\alpha\}}{10} = [\alpha] + \frac{[10\{\alpha\}]}{10} + \frac{\{10\{\alpha\}\}}{10} = \\ &= [\alpha] + \frac{[10\{\alpha\}]}{10} + \frac{10\{10\{\alpha\}\}}{10^2} = \dots \end{aligned} \tag{3}$$

and so one. Now, if we put down the integer $[\alpha]$ in a decimal representation:

$$[\alpha] = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \dots + a_0$$

and we denote

$$a_{-1} = [10\{\alpha\}],\ a_{-2} = [10\{10\{\alpha\}\}],\ \dots$$

we get the length α in a decimal form (1).

On the other hand, we can construct an abscissa CD to each decimal number α in the form (1) so that we get this number α from its length by algorithm (3): Firstly, we construct an abscissa of the length $a_n a_{n-1} \dots a_0$ with the help of an unit abscissa AB; then we complete it with an abscissa which is a_{-1}-multiple of an abscissa which length is equal to $\frac{1}{10}$ of the length AB, and so one.

Exercises

1. Apply algorithm (3) to concrete examples; especially to measuring a diagonal in an unit square.

2. Find a corresponding length to a given number which has a finite decimal expansion.

(b) **Decimal expansions of rational numbers.** We can directly use the algorithm (3) for an expansion of a rational number $\alpha = \frac{p}{q}$ from this canonical form, if we denote by $[\alpha]$ the greatest integer which is not greater than α and by $\{\alpha\}$ the fractional part of α, i.e. $\{\alpha\} = \alpha - [\alpha]$. From that we have

$$\frac{p}{q}=\left[\frac{p}{q}\right]+\left\{\frac{p}{q}\right\}=\left[\frac{p}{q}\right]+\frac{10\left\{\frac{p}{q}\right\}}{10}=\left[\frac{p}{q}\right]+\frac{\left[10\left\{\frac{p}{q}\right\}\right]}{10}+\frac{\left\{10\left\{\frac{p}{q}\right\}\right\}}{10}=$$
$$=\left[\frac{p}{q}\right]+\frac{\left[10\left\{\frac{p}{q}\right\}\right]}{10}+\frac{\left[10\left\{10\left\{\frac{p}{q}\right\}\right\}\right]}{10^2}+\ldots=a_na_{n-1}\ldots a_0,\ a_{-1}a_{-2}\ldots=\alpha. \tag{4}$$

It is an extension of a well-known algorithm of division $p{:}q$ of two integers with digits which are standing on decimal places. As the fractional parts

$$\left\{\frac{p}{q}\right\},\ \left\{10\left\{\frac{p}{q}\right\}\right\},\ \left\{10\left\{10\left\{\frac{p}{q}\right\}\right\}\right\},\ \ldots$$

which appear in (4) can only be of the form

$$\frac{0}{q},\ \frac{1}{q},\ \frac{2}{q},\ \ldots,\ \frac{q-1}{q}.$$

They have to repeat, therefore the decimal expansion (4) will be periodic. We can abridge it and write α in the representation

$$\alpha=a_na_{n-1}\ldots a_0,\ a_{-1}a_{-2}\ldots a_{-k}\overline{a_{-(k+1)}\ldots a_{-(k+m)}} \tag{5}$$

where the last m digits present a period and $a_{-1}a_{-2}\ldots a_{-k}$ is a foreperiod.

On the other hand, given any periodic decimal expansion (5), there exists a rational number $\frac{p}{q}$ in a canonical form which can be obtained by using $\frac{p}{q}$ in the algorithm (3). We calculate it from the following equations:

$$\begin{aligned}\frac{p}{q}&=\frac{1}{10^k}\frac{p_0}{q_0}+a_na_{n-1}\ldots a_0,\ a_{-1}a_{-2}\ldots a_{-k}\\ \frac{p_0}{q_0}&=\frac{p_0}{q_0}10^m-a_{-(k+1)}a_{-(k+2)}\ldots a_{-(k+m)}.\end{aligned} \tag{6}$$

The last equation presents the calculation of $\frac{p_0}{q_0}$ with a pure-periodic expansion $0,\ \overline{a_{-(k+1)}a_{-(k+2)}\ldots a_{-(k+m)}}$.

By section (a), each decimal expansion represents some length and in particular a non-periodic expansion represents the length of some abscissa which cannot be expressed by a rational number. This was the main reason for completing the rational numbers with the other numbers, i.e. irrational numbers, which are identic with non-periodic expansions in our point of view. The diagonal of a unit square was the first well-known abscissa with that properties.

Euclid showed the proof in his "Principia" in the following way: If a diagonal of a unit square has a length which equals a rational number $\frac{p}{q}$, then it would be $\left(\frac{p}{q}\right)^2 = 2$ according to the Pythagoras theorem. But this equation has no solution in rational numbers. This follows from the theorem on the unique decomposition of an integer into a product of primes (Euclid showed that using his algorithm).

This result caused a crisis in the atomic concept of the space, especially of the length. In fact, if the unit abscissa AB included q atoms and another abscissa CD included p atoms then the length of abscissa CD would be expressed in a form $\frac{p}{q}$ by using the unit abscissa AB.

All our numerical calculus is performed in a finite time and by using rational numbers only. From that point of view, the extention of rational numbers by non-periodic numbers (i.e. not finite expansions) seems to be forced by a philosophy of mathematics only and to have no practical importance. The contrary is true. For example: The irrational number π acts in the calculus of a circulating course of satellites. The more complex experiments we make, the more precisely we need to know the number π. We would never write it in the whole, but we need to know a method-algorithm using which we can evaluate the next term in its decimal expansion whenever we need it.

Also, to know an irrational number means to know an algorithm by which we can evaluate any digits in its decimal expansion. We identify that number with its algorithm.

Exercises

1. Exercise the expansion of a rational number given in a canonical form into its decimal representation and verify

(i) the decimal expansion of a number $\frac{p}{q}$ is pure-periodic and infinite if the denominator q has no common divisor with a number 10;

(ii) the decimal expansion of a number $\frac{p}{q}$ is finite if $q = 2^\alpha \cdot 5^\beta$;

(iii) the decimal expansion of a number $\frac{p}{q}$ is infinite with a foreperiod if the denominator q is divided by 2 or 5 and also if q is divided by a number which is not divided by 2 or 5.

2. Exercise algorithm (3) in which the number 10 is substituted by 2 (i.e. it means an expression in the binary system).

3. Exercise calculating a rational number back in a canonical form from a given periodical expansion (6).

4. Construct non-periodical expansions, for example

$$a_{-k} = \begin{cases} 1 \text{ if } k \text{ is a square} \\ 0 \text{ in other casses.} \end{cases}$$

5. Prove the irrationality of $\sqrt[3]{2}$ by using the Euclidean method, similarly for $\sqrt{2} + \sqrt{3}$, $\sqrt{2} + \sqrt{3} + \sqrt{5}$.

6. Exercise the following Estermann's proof of the irrationality of $\sqrt{2}$ in which the fundamental theorem of arithmetics is not used for other roots.

Let $\sqrt{2}$ be a ratinal number. Then a set A of all positive integers k such that $k\sqrt{2}$ is a positive integer is not empty and then it has a minimum and we denote it k_0. Then we have

(i) $k_0\sqrt{2} - k_0$ is a positive integer,

(ii) $k_0\sqrt{2} - k_0 \in A$ because $\sqrt{2}(k_0\sqrt{2} - k_0) = 2k_0 - \sqrt{2}k_0$ is again a positive integer,

(iii) $k_0\sqrt{2} - k_0 < k_0$ which is a contradiction with the fact that k_0 is the minimum of the set A.

(c) **Ordering.** Two real numbers α, β in the decimal representation

$$\alpha = a_n a_{n-1} \ldots a_0, a_{-1} a_{-2} \ldots$$

$$\beta = b_m b_{m-1} \ldots b_0, b_{-1} b_{-2} \ldots$$

fulfil the inequality $\alpha < \beta$ if it is not the case (2) and if it is true either $m > n$ or $m = n$ and

$$a_n = b_n, a_{n-1} = b_{n-1}, \ldots, a_{n-i} = b_{n-i}, a_{n-i-1} < b_{n-i-1}$$

i.e. it is defined by an inequality between first digits in which they are different. For example:

$$0{,}000\ldots 0 a_{-k} a_{-k-1} \ldots \leqq \frac{1}{10^{k-1}} = 0{,}000\ldots 010\ldots$$

The above definition of rational numbers $\frac{p}{q}$, $\frac{r}{s}$ is in accordance with the definition of $\frac{p}{q} < \frac{r}{s}$ by the inequality

$$ps < qr$$

therefore if we develop $\frac{p}{q}$, $\frac{r}{s}$ into a decimal form by (4) then $\left[\frac{p}{q}\right] = \left[\frac{r}{s}\right]$ implies

$\left\{\dfrac{p}{q}\right\} < \left\{\dfrac{r}{s}\right\}$, i.e. $10\left\{\dfrac{p}{q}\right\} < 10\left\{\dfrac{r}{s}\right\}$. If $\left[10\left\{\dfrac{p}{q}\right\}\right] = \left[10\left\{\dfrac{r}{s}\right\}\right]$ then $\left\{10\left\{\dfrac{p}{q}\right\}\right\} <$ $< \left\{10\left\{\dfrac{r}{s}\right\}\right\}$ and so on, then there must be a strong inequality somewhere between whole parts of real numbers.

The number 0 has a decimal expansion

$$0{,}000\ldots0\ldots$$

and therefore $0 < \alpha$ is true for all real numbers $\alpha \neq 0$ in a decimal form (1). We add all negative real numbers to the positive real numbers in the same way as for rational numbers: we attach the sign $-$ to each positive real number and their ordering will be inverse, i.e. if $\alpha < \beta$ then $-\alpha > -\beta$.

We speak about a number axis if we draw a line in a plane and a point 0 on it, on the right side of 0 we can depict lenghts which are corresponding to positive real numbers, on the left side of 0 those corresponding to negative real numbers. Let $|\alpha - \beta|$ denote the distance between two points on a number axis.

If α, β have the same digits up to the place of order k then

$$|\alpha - \beta| \leqq \frac{1}{10^k} \tag{7}$$

since $|\alpha - \beta|$ has all digits (as we shall see) equal to zero up to the place of order k.

On the other hand, if (7) is true then real numbers α, β need not have the same digits up to the place of order k because (if $a_{-j} < 9$)

$$\begin{aligned} &a_n a_{n-1}\ldots a_0,\ a_{-1}a_{-2}\ldots a_{-j}999\ldots 9a_{-(k+1)}a_{-(k+2)}\ldots + \frac{1}{10^k} = \\ &= a_n a_{n-1}\ldots a_0,\ a_{-1}a_{-2}\ldots(a_{-j}+1)000\ldots 0a_{-(k+1)}a_{-(k+2)}\ldots \end{aligned} \tag{8}$$

By (8), if we know the digits of β and we know that (7) is true for α, then α has the same digits as β up to the digit located before a digit followed by all 9's or all 0's up to the digit of order k.

Addition $\alpha + \beta$ and a substraction $\alpha - \beta$ which were used above can be performed also in the following way which is different from the definition in (e).

To begin with we add and substract individual digits (we complet zeros in order that $n = m$)

$$(a_n + b_n)(a_{n-1} + b_{n-1})\ldots(a_0 + b_0),\ (a_{-1} + b_{-1})(a_{-2} + b_{-2})\ldots$$
$$(a_n - b_n)(a_{n-1} - b_{n-1})\ldots(a_0 - b_0),\ (a_{-1} - b_{-1})(a_{-2} - b_{-2})\ldots$$

If $a_{-k} + b_{-k} \geqq 10$ then we make the carry yielding

$$(a_{-(k-1)} + b_{-(k-1)} + 1) \quad \text{and} \quad (a_{-k} + b_{-k} - 10);$$

if $a_{-k} - b_{-k} > 0$ then we make the carry yielding

$$(a_{-(k-1)} - b_{-(k-1)} - 1) \quad \text{and} \quad (a_{-k} - b_{-k} + 10).$$

(If all preceding digits $(a_{-k} - b_{-k})$ are zeros then we only transfer a sign — at the beginning of a number.)

Using the absolute value $|\alpha - \beta|$, we remove the sign in $\alpha - \beta$. It permits us to treat two possible situations $\alpha < \beta$ and $\beta < \alpha$ simultaneously.

Exercises

1. Find decimal expansions of concrete rational numbers $\frac{p}{q}, \frac{r}{s}$ and verify an inequality between them using those expansions.

2. Show that we can insert a third number between two arbitrary distinct real numbers given by decimal expansions.

3. Exercise the properties of an absolute value, especially the triangle-inequality $|\alpha + \beta| \leqq |\alpha| + |\beta|$.

4. What initial digits has a number α if we know that it is true

$$|\alpha - 0{,}059109999976| \leqq \frac{1}{10^9}$$

5. Add two real numbers:

$$0{,}90990999099990\ldots$$
$$1{,}10010001000010\ldots.$$

(d) **Limit and supremum.** Let us have a sequence of real numbers $\alpha(i)$, $i = 1, 2, 3, \ldots$ and a real number α in a decimal form

$$\alpha(i) = a_{n(i)}(i)a_{n(i)-1}(i)\ldots a_0(i),\ a_{-1}(i)a_{-2}(i)\ldots a_{-k}(i)\ldots$$
$$\alpha = a_n a_{n-1}\ldots a_0,\ a_{-1}a_{-2}\ldots a_{-k}\ldots.$$

We say that $\alpha(i)$ converges to α ($\alpha(i) \to \alpha$) or α is a limit of a sequence $\alpha(i)$

$$\lim_{i \to \infty} \alpha(i) = \alpha$$

if $n(i) = n$ for all sufficiently large i and if one of the two following casses occurs:

(j) α is not in the form (2) and

$$a_n(i) \to a_n,\ a_{n-1}(i) \to a_{n-1},\ \ldots,\ a_0(i) \to a_0,\ a_{-1}(i) \to a_{-1},\ a_{-2}(i) \to a_{-2},\ \ldots$$

while the sequence of nonnegative integers is convergent and if it is constant after finitely many terms; or (jj) α is in the form (2) and the sequence $\alpha(i)$ can be expressed as consisting of at most two parts, the firts part converging by (j) to α which is written in a decimal form (2) with terminal digits equal to zero, and the (eventual) second part converging by (j) to α which is written in a decimal form (2) with terminal digits equal to 9.

For example, if

$$\alpha(i) = \frac{1}{10^i} = 0{,}000\ldots010\ldots$$

then $\alpha(i) \to 0$. Or if

$$\alpha(1) = 1{,}100\ldots$$
$$\alpha(2) = 0{,}900\ldots$$
$$\alpha(3) = 1{,}010\ldots$$
$$\alpha(4) = 0{,}990\ldots$$

and so on, then $\alpha(i) \to 1$.

Let $A = \{\alpha(t);\ t \in T\}$ be a set of real numbers in a decimal form

$$\alpha(t) = a_{n(t)}(t)a_{n(t)-1}(t)\ldots a_0(t),\ a_{-1}(t)a_{-2}(t)\ldots a_{-k}(t)\ldots$$

and let it be upper bounded by a number β. Then we can find the smallest upper bound, called the supremum of a set A, in a following way:

$\alpha(t) \leqq \beta$ implies that non-negative integers $n(t)$ are upper bounded. We ought to find $n = \max n(t)$.

For numbers $\alpha(t)$ for which $n(t) = n$, we shall find $a_n = \max a_n(t)$; for numbers $\alpha(t)$ for which $n(t) = n$, $a_n(t) = a_n$ we shall find $a_{n-1} = \max a_{n-1}(t)$ and so on. Summarizing,

$$\sup A = \alpha = a_n a_{n-1}\ldots a_0,\ a_{-1}a_{-2}\ldots a_{-k}\ldots$$

since the first digit in which $\alpha(t)$ and α differ is greater in α and there always exists $\alpha(t)$ which has arbitrarily many initial digits in common with α.

For instance, if we choose for $t > 0$

$$\alpha(t) = 0,\ a_{-1}(t)a_{-2}(t)\ldots a_{-k}(t)\ldots$$

$$a_{-k}(t) = \begin{cases} [t] \pmod{10} \text{ if } k \text{ is odd} \\ [kt + k] \pmod{10} \text{ if } k \text{ is even} \end{cases}$$

(where $a \pmod{10}$ denotes the remainder after a division by 10.) then

$$a_{-1}(t) = 9 \quad \text{for} \quad 9 + m10 \leqq t < 10 + m10, \qquad m = 0, 1, 2, \ldots$$

$$a_{-2}(t) = 1 \quad \text{for} \ \frac{19}{2} + m10 \leqq t < 10 + m10$$

$$a_{-3}(t) = 9 \quad \text{for } \frac{19}{2} + m10 \leqq t < 10 + m10$$

$$a_{-4}(t) = 3 \quad \text{for } \frac{39}{4} + m10 \leqq t < 10 + m10 \ldots$$

i.e. a supremum equals

$$\alpha = 0{,}9193\ldots.$$

Similarly we find the greatest lower bound, called the infimum.

Exercises

1. Let a sequence $\alpha(i)$, $i = 1, 2, 3, \ldots$ be defined in the following way

$$\alpha(i) = 0,\ a_{-1}(i)a_{-2}(i)\ldots a_{-k}(i)\ldots$$

$$a_{-k}(i) = \left[\frac{k^2i^2 + ki + 1}{3ki^2 + 2}\right] \pmod{10}$$

Find a limit of $\alpha(i)$.

2. Let a set $A = \{\alpha(t);\ t > 0\}$ be defined in the following way

$$\alpha(t) = 0,\ a_{-1}(t)a_{-2}(t)\ldots a_{-k}(t)\ldots$$

$$a_{-k}(t) = [k^2t^2] \pmod{10}.$$

Find sup A.

(e) **Operations.** Assign a rational number

$$\alpha_k = a_na_{n-1}\ldots a_0,\ a_{-1}a_{-2}\ldots a_{-k}000$$

to a number α which has a decimal expansion (1); we will call it the initial k-section. A number α is given if there is given an algorithm by which can compute any of its digits. We only recognize a finite number of its digits, i.e. only α_k in a finite time (up to exceptions). Then we can define operations $\alpha + \beta$, $\alpha - \beta$, $\alpha \cdot \beta$, $\frac{\alpha}{\beta}$ between real numbers α, β given in a decimal representation using their initial sections only and it will be in a form $\alpha_k + \beta_k$, $\alpha_k - \beta_k$, $\alpha_k \cdot \beta_k$, $\frac{\alpha_k}{\beta_k}$. They are well-known operations between rational numbers. We can write those numbers in a decimal form and inquire about sequences which are formed by those numbers for $k = 1, 2, 3, \ldots$ if they converge by (d). Really, in a university course of mathematics we can prove that there exist limits of those sequences and that passing to the limit all properties of operations with rational numbers are transfered to real numbers. If we stand on a position that we have a non-limited time available, i.e. that we have enumerated $\alpha + \beta$, $\alpha - \beta$, $\alpha \cdot \beta$, $\frac{\alpha}{\beta}$ without a rest then we can estimate the differences between those numbers and $\alpha_k + \beta_k$, $\alpha_k - \beta_k$, $\alpha_k \cdot \beta_k$, $\frac{\alpha_k}{\beta_k}$ using the distributive law and the triangle-inequality which are well-known properties of rational numbers.

We get (using (7))

$$|\alpha + \beta - (\alpha_k + \beta_k)| \leqq |\alpha - \alpha_k| + |\beta - \beta_k| \leqq \frac{2}{10^k}$$

$$|\alpha - \beta - (\alpha_k - \beta_k)| \leqq |\alpha - \alpha_k| + |\beta - \beta_k| \leqq \frac{2}{10^k}$$

$$\begin{aligned} |\alpha\beta - \alpha_k\beta_k| &= |\alpha\beta - \alpha\beta_k + \alpha\beta_k - \alpha_k\beta_k| = |\alpha(\beta - \beta_k) + \beta_k(\alpha - \alpha_k)| \leqq \\ &\leqq |\alpha|\,|\beta - \beta_k| + |\beta_k|\,|\alpha - \alpha_k| \leqq \frac{1}{10^k}\left(|\alpha_{k_0}| + |\beta_{k_0}| + \frac{2}{10^{k_0}}\right) \end{aligned} \tag{9}$$

$$\begin{aligned} \left|\frac{\alpha}{\beta} - \frac{\alpha_k}{\beta_k}\right| &= \left|\frac{\alpha}{\beta} - \frac{\alpha}{\beta_k} + \frac{\alpha}{\beta_k} - \frac{\alpha_k}{\beta_k}\right| = \left|\alpha\left(\frac{1}{\beta} - \frac{1}{\beta_k}\right) + \frac{1}{\beta_k}(\alpha - \alpha_k)\right| \leqq \\ &\leqq |\alpha| \frac{|(\beta - \beta_k)|}{|\beta\beta_k|} + \frac{1}{|\beta_k|}|\alpha - \alpha_k| \leqq \frac{1}{10^k}\left(\frac{|\alpha_{k_0}| + |\beta_{k_0}| + \frac{1}{10^{k_0}}}{|\beta_{k_0}|^2}\right) \end{aligned}$$

where k_0 is an arbitrary positive integer for which $\beta_{k_0} \neq 0$, $k_0 \leqq k$. The right sides of the above inequalities represent estimates of errors with which the real numbers $\alpha + \beta$, $\alpha - \beta$, $\alpha \cdot \beta$, $\frac{\alpha}{\beta}$ are approximated by rational numbers $\alpha_k + \beta_k$, $\alpha_k - \beta_k$, $\alpha_k \cdot \beta_k$, $\frac{\alpha_k}{\beta_k}$. Using them, we can solve exercises of following type: Find k to a given n so that

$$\left|\frac{\alpha}{\beta} - \frac{\alpha_k}{\beta_k}\right| \leqq \frac{1}{10^n}$$

i.e. find a rational number which approximates $\frac{\alpha}{\beta}$ with error less than or equal to $\frac{1}{10^n}$. Then by (8) a real number $\frac{\alpha}{\beta}$ has the same digits as $\frac{\alpha_k}{\beta_k}$ up to a digit which is located before a digit followed by all 0's or all 9's up to the digit or order n. Thus there can be many non-coinciding digits up to the order n. In a numerical calculation, the precision depends on the exactness with which we substitute one number for another and not on the digits being identical. We come to a situation when it is more convenient to consider a real number α given if we know how to calculate rational numbers $\frac{p}{q}$ in a canonical form which approximate α with an arbitrary exactness; the calculation of its digits themselves is seen not to be important.

Exercises

1. Let

$$\alpha = 0,\ a_{-1}a_{-2}\ldots a_{-k}\ldots$$

$$\beta = 0,\ b_{-1}b_{-2}\ldots b_{-k}\ldots$$

where

$$a_{-k} = \begin{matrix} 1 \text{ if } k = n^2 \\ 0 \text{ in other casses} \end{matrix}$$

$$b_{-k} = \begin{matrix} 9 \text{ if } k = n^3 \\ 0 \text{ in other casses} \end{matrix}$$

Find a rational number in a canonical form which approximates $\frac{\alpha}{\beta}$ with an error less than $\frac{1}{10^9}$.

2. Transfer two rational numbers $\frac{p}{q}, \frac{r}{s}\left(\frac{p}{q}, \frac{r}{s} < 1\right)$ from a canonical form to a decimal form,

$$\frac{p}{q} = 0,\ a_{-1}a_{-2}\ldots a_{-k}\ldots$$

$$\frac{r}{s} = 0,\ b_{-1}b_{-2}\ldots b_{-k}\ldots$$

Perform a formal addition of digits

$$0,\ (a_{-1} + b_{-1})(a_{-2} + b_{-2})\ldots(a_{-k} + b_{-k})\ldots$$

From it you can directly find a decimal expansion of their sum

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs}$$

3. We can introduce the operations between α and β geometrically by constructing corresponding abscissas of lengths α, β to them by (a) and then we construct abscissas with lengths $\alpha + \beta$, $\alpha - \beta$, $\alpha \cdot \beta$, $\frac{\alpha}{\beta}$ and we can express their

lengths back in a decimal representation by (a). The more exactness we would like to achieve the larger unit of length we must choose. Then we need a non--limited space in addition to a non-limited time. The error of this geometrical method depends on many factors, its estimate cannot be made in the same simple form as in a numerical calculus.

We have to carry out experimentaly a geometrical construction of $\alpha\beta$ and $\frac{\alpha}{\beta}$, to calculate their decimal form and to verify how the exactness of the calculation will be changed if we magnify the unit of length.

4. The relation between real and rational numbers will be preserved by a transfer of operations. In particular:

$$\text{If } \alpha < \beta \text{ then } \alpha + \gamma < \beta + \gamma$$

$$\text{if } \alpha < \beta \text{ and } \gamma > 0 \text{ then } \alpha\gamma < \beta\gamma.$$

Using these two properties, we have to show that (here k, n are positive integers)

(i) if $0 < \alpha < \beta$ then $\alpha^n < \beta^n$,

(ii) if $0 < \alpha < 1$ and $k < n$ then $\alpha^k > \alpha^n$.

(f) **Functions.** By a function we shall understand an arbitrary sequence of operations (called an algorithm), we can sign them $f(x)$ comprehensively so that if we give a real number α at the beginning then we can calculate a real number $f(\alpha)$ unambiguously by those operations. We can do it provided that we have a non-limited time at our disposal.

For example:

$$f(x) = 3x^2 + 2x + 1 \tag{10}$$

or

$$f(x) = \text{we change each even digit in a decimal expansion of } x \text{ to zero.} \tag{11}$$

In a general case, we can find a real number α in a limited time only by calculating a rational number $\frac{p}{q}$ for each positive integer k so that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{10^k}.$$

In the general case, the imput x may be a rational number $\frac{p}{q}$. If the number of operations in $f(x)$ is infinite then again we can find a real number $f\left(\frac{p}{q}\right)$ just by computing a rational number $\frac{P}{Q}$ for each k such that it is true:

$$\left|f\left(\frac{p}{q}\right) - \frac{P}{Q}\right| \leqq \frac{1}{10^k}. \tag{12}$$

In order to know the real number $f(\alpha)$ in the above sense, we have to find an estimate

$$\left|f(\alpha) - f\left(\frac{p}{q}\right)\right| < c \cdot \left|\alpha - \frac{p}{q}\right| \tag{13}$$

using the algorithm $f(x)$, where c does not depend on k, $\frac{p}{q}$ and it can be calculated from rational estimations

$$\frac{p_0}{q_0} < \alpha < \frac{p_1}{q_1}$$

by using some elementary method.

Summarizing we get

$$\left|f(\alpha) - \frac{P}{Q}\right| \leqq \left|f(\alpha) - f\left(\frac{p}{q}\right)\right| + \left|f\left(\frac{p}{q}\right) - \frac{P}{Q}\right| \leqq \frac{c+1}{10^k}$$

We say we have found $f(\alpha)$ with an exactness $\frac{c+1}{10^k}$ or that a rational number $\frac{P}{Q}$ approximates $f(\alpha)$ with an error which is not greater than $\frac{c+1}{10^k}$.

For example: we can get

$$\left|f(\alpha) - f\left(\frac{p}{q}\right)\right| = \left|3\alpha^2 + 2\alpha + 1 - \left(3\left(\frac{p}{q}\right)^2 + 2\left(\frac{p}{q}\right) + 1\right)\right| =$$

$$= \left|3\left(\alpha + \frac{p}{q}\right)\left(\alpha - \frac{p}{q}\right) + 2\left(\alpha - \frac{p}{q}\right)\right| = \left|3\left(\alpha + \frac{p}{q}\right) + 2\right| \cdot \left|\left(\alpha - \frac{p}{q}\right)\right| \leqq$$

$$\leqq \left|3\left(2\alpha_{k_0} + \frac{3}{10^{k_0}}\right)\right| \cdot \left|\alpha - \frac{p}{q}\right|$$

for a function (10) where we have obtain the last inequality from the fact that if $k_0 \leqq k$ then

$$\alpha \leqq \alpha_{k_0} + \frac{1}{10^{k_0}}$$

$$\frac{p}{q} < \alpha + \frac{1}{10^{k_0}} \leqq \alpha_{k_0} + \frac{1}{10^{k_0}} + \frac{1}{10^{k_0}}$$

Therefore, in the sequel we shall consider a function $f(x)$ given if we know how to calculate (12) and (13). Thus $f\left(\frac{p}{q}\right) \to f(\alpha)$ is implied by the convergence $\frac{p}{q} \to \alpha$ for those functions. We call them continous functions. The function (11) does not belong among such functions since $f(1) = 0$ and $f(0{,}999\ldots) = = 0{,}9090\ldots$.

Root. At the beginning, our situation is so that we do not know $f(x)$ and we have only a given property which $f(x)$ ought to fulfil. For example: Given a positive integer s, by a function $x^{\frac{1}{s}}$ called the s-root we shall understand such a function $f(x)$ defined for $\alpha > 0$ for which the output $f(\alpha) = \alpha^{\frac{1}{s}}$ has so property that

$$\alpha^{\frac{1}{s}} \cdot \alpha^{\frac{1}{s}} \ldots \alpha^{\frac{1}{s}} = \left(\alpha^{\frac{1}{s}}\right)^s = \alpha \tag{14}$$

If we know that such a number $\alpha^{\frac{1}{s}}$ exists we can compute (13) in the following way:

$$\left|\alpha^{\frac{1}{s}} - \left(\frac{p}{q}\right)^{\frac{1}{s}}\right| = \frac{\left|\alpha - \frac{p}{q}\right|}{\left(\alpha^{\frac{1}{s}}\right)^{s-1} + \left(\alpha^{\frac{1}{s}}\right)^{s-2} \cdot \left(\frac{p}{q}\right)^{\frac{1}{s}} + \ldots + \alpha^{\frac{1}{s}}\left(\left(\frac{p}{q}\right)^{\frac{1}{s}}\right)^{s-2} + \left(\left(\frac{p}{q}\right)^{\frac{1}{s}}\right)^{s-1}}$$

where $\alpha^{\frac{1}{s}}, \left(\frac{p}{q}\right)^{\frac{1}{s}}$ can be estimated from below by finding a number k_1 for which $10^{-k_1 s} \leqq \alpha$ $\left(\text{that is, } 10^{-k_1} \leqq \alpha^{\frac{1}{s}}\right)$ and if

$$\left|\alpha - \frac{p}{q}\right| \leqq \frac{1}{10^k}$$

then

$$\frac{1}{10^{k_3 s}} \leqq \frac{1}{10^{k_1 s}} - \frac{1}{10^{k_2}} \leqq \alpha - \frac{1}{10^k} \leqq \frac{p}{q}$$

where $k_1 s < k_2 \leqq k$. Using these estimates we can calculate the constant c in (13), for example in the form

$$c = \frac{10^{k_3(s-1)}}{s}.$$

From (14) it follows that

$$\left(\frac{p}{q}\right)^{\frac{1}{s}} = \frac{p^{\frac{1}{s}}}{q^{\frac{1}{s}}} \tag{15}$$

Therefore, in order to be able to compute $x^{\frac{1}{s}}$ we need only to know how to calculate s-roots of positive integers.

If we know that there exists a real number $a = p^{\frac{1}{s}}$ with a property $a^s = p$ (what is proved in a university course of Calculus) then we can compute the coefficients in its decimal expansion in the following way:

By the fundamental algorithm (3) to express a number in a decimal form (1) is the same as to calculate the integer parts

$$a_{-k} = [10\{10\{\ldots 10\{10\{a\}\}\ldots\}\}].$$

For a number from the right-hand side it is true

$$\begin{aligned}10\{10\{\ldots 10\{10\{a\}\}\ldots\}\} &= 10\{10\{\ldots 10\{10(a - j_1)\}\ldots\}\} = \\ &= 10\{10\{\ldots 10(10a - 10j_1 - j_2)\ldots\}\} = 10^k a - j^*\end{aligned}$$

To find the integer part of $10^k a - j$ means to solve the inequality

$$a \leqq 10^k a - j < a + 1 \tag{16}$$

with respect to the integer a.

Applying algorithm (3), we can construct some real numbers a with any special property if we know that such a number exists and if we can rewrite (16) (using this property) in a form in which the unknown a does not appear. So for example: if $a = p^{\frac{1}{s}}$ then we can reduce (16) to the form

$$a \leqq 10^k p^{\frac{1}{s}} - j < a + 1$$

$$a + j \leqq 10^k p^{\frac{1}{s}} < a + j + 1$$

$$(a + j)^s \leqq 10^{ks} p < (a + j + 1)^s$$

We can solve the last inequality trying a substitution for $a = 0, 1, 2, \ldots, 9$. For example, for $a = \sqrt[3]{2}$ we get

$$\begin{aligned}\sqrt[3]{2} &= [\sqrt[3]{2}] + \{\sqrt[3]{2}\} = 1 + (\sqrt[3]{2} - 1) = 1 + \frac{10(\sqrt[3]{2} - 1)}{10} = \\ &= 1 + \frac{[10(\sqrt[3]{2} - 1)]}{10} + \frac{\{10(\sqrt[3]{2} - 1)\}}{10} = 1 + \frac{2}{10} + \frac{10(\sqrt[3]{2} - 1) - 2}{10} = \\ &= 1 + \frac{2}{10} + \frac{10(10\sqrt[3]{2} - 12)}{10^2} = 1 + \frac{2}{10} + \frac{[10^2\sqrt[3]{2} - 120]}{10^2} +\end{aligned}$$

* Exactly $a_{-k} = [10^k a - [10^{k-1} a]]$.

$$+\frac{\{(10^2\sqrt[3]{2}-120)\}}{10^2}=1+\frac{2}{10}+\frac{5}{10^2}+\frac{10(10^2\sqrt[3]{2}-125)}{10^3}=$$

$$=1+\frac{2}{10}+\frac{5}{10^2}+\frac{9}{10^3}+\frac{10(10^3\sqrt[3]{2}-1259)}{10^4}$$

We solve inequality (12) for a function $f(x)=x^{\frac{1}{s}}$ by finding a number k_1 for a number using the estimate (9) so that the inequality

$$\left|\frac{p^{\frac{1}{s}}}{q^{\frac{1}{s}}}-\frac{\left(p^{\frac{1}{s}}\right)_{k_1}}{\left(q^{\frac{1}{s}}\right)_{k_1}}\right|\leqq\frac{1}{10^k}$$

be implied by the inequalities

$$\left|p^{\frac{1}{s}}-\left(p^{\frac{1}{s}}\right)_{k_1}\right|\leqq\frac{1}{10^{k_1}},\quad\left|q^{\frac{1}{s}}-\left(q^{\frac{1}{s}}\right)_{k_1}\right|\leqq\frac{1}{10^{k_1}}.$$

Then the obtained rational number $\frac{P}{Q}$ will be equal to quotient of the initial k_1-sections of real numbers $p^{\frac{1}{s}}$ and $q^{\frac{1}{s}}$ which we denote by $\left(p^{\frac{1}{s}}\right)_{k_1}$, $\left(q^{\frac{1}{s}}\right)_{k_1}$.

Exponential. Using $x^{\frac{1}{s}}$, we can construct a power function x^β and an exponential function β^x, i.e. we can calculate a general power α^β $(\alpha>0)$ if we find an estimate

$$\left|\alpha^\beta-\left(\frac{p}{q}\right)^{\frac{r}{s}}\right|$$

in terms of

$$\left|\alpha-\frac{p}{q}\right|,\quad\left|\beta-\frac{r}{s}\right|,$$

because, according to what has been said, we can calculate

$$\left(\frac{p}{q}\right)^{\frac{r}{s}}=\frac{(p^r)^{\frac{1}{s}}}{(q^r)^{\frac{1}{s}}}$$

Firstly, we shall estimate $\left(\text{similarly as for } x^{\frac{1}{s}}\right)$ by usual algebraic reductions

$$\left|\alpha^{\frac{r}{s}}-\left(\frac{p}{q}\right)^{\frac{r}{s}}\right|=\left|\alpha-\frac{p}{q}\right|\cdot\frac{\dfrac{\left|\left(\alpha^{\frac{1}{s}}\right)^r-\left(\left(\frac{p}{q}\right)^{\frac{1}{s}}\right)^r\right|}{\left|\alpha^{\frac{1}{s}}-\left(\frac{p}{q}\right)^{\frac{1}{s}}\right|}}{\dfrac{\left|\left(\alpha^{\frac{1}{s}}\right)^s-\left(\left(\frac{p}{q}\right)^{\frac{1}{s}}\right)^s\right|}{\left|\alpha^{\frac{1}{s}}-\left(\frac{p}{q}\right)^{\frac{1}{s}}\right|}}\leqq c_1\left|\alpha-\frac{p}{q}\right|$$

where we compute the constant c_1 by estimating $\alpha^{\frac{1}{s}}$ from below and from above and by using the $\left|\alpha-\frac{p}{q}\right|\leqq\frac{1}{10^k}$ to estimate $\left(\frac{p}{q}\right)^{\frac{1}{s}}$ from below and from above too. Secondly, we estimate

$$\left|\alpha^{\beta}-\alpha^{\frac{r}{s}}\right|=\alpha^{\bar{\beta}}|\log\alpha|\left|\beta-\frac{r}{s}\right|\leqq c_2\left|\beta-\frac{r}{s}\right|$$

using the derivative $(\alpha^x)'=\alpha^x\cdot\log\alpha$ and the Lagrange's mean-value theorem where $\bar{\beta}$ is a suitable number between β and $\frac{r}{s}$. Here it is necessary to employ results from a university course of Calculus. Finally, we use

$$\left|\alpha^{\beta}-\left(\frac{p}{q}\right)^{\frac{r}{s}}\right|\leqq\left|\alpha^{\beta}-\alpha^{\frac{r}{s}}\right|+\left|\alpha^{\frac{r}{s}}-\left(\frac{p}{q}\right)^{\frac{r}{s}}\right|$$

Logarithm. A logarithmic function to the base 10 for $\alpha>0$ is defined by the property

$$10^{\log_{10}\alpha}=\alpha. \tag{17}$$

Using the derivative $(\log_{10}x)'=\frac{1}{x}\cdot\log_{10}e$ and the Lagrange's mean-value theorem which is inevitable we get (13) in the form

$$\left|\log_{10}\alpha-\log_{10}\frac{p}{q}\right|=\frac{1}{\bar{\alpha}}\cdot\log_{10}e\left|\alpha-\frac{p}{q}\right|\leqq\frac{1}{\alpha_{k_0}-\dfrac{1}{10^{k_1}}}\left|\alpha-\frac{p}{q}\right|$$

where $\bar{\alpha}$ is a suitable number between α and $\frac{p}{q}$ and k_0, k_1 are such integers for which $\alpha_{k_0}>0$, $k_0<k_1\leqq k$.

From equality (17) it follows

$$\log_{10} \frac{p}{q} = \log_{10} p - \log_{10} q$$

hence it is sufficient to know how to calculate the logarithms of integers. Here we can use once more algorithm (3), because if $a = \log_{10} p$, i.e. $10^a = p$, then (16) reduces to

$$a \leqq 10^k \log_{10} p - j < a + 1$$

$$a + j \leqq 10^k \log_{10} p < a + j + 1$$

$$10^{a+j} \leqq p^{10^k} < 10^{a+j+1}$$

where the last inequality can be solved experimentally substituting $a = 0, 1, 2, \ldots, 9$. For instance for $a = \log_{10} 2$ we get

$$\log_{10} 2 = 0 + \frac{10 \log_{10} 2}{10} = 0 + \frac{[10 \log_{10} 2]}{10} + \frac{\{10 \log_{10} 2\}}{10} =$$

$$= 0 + \frac{3}{10} + \frac{10(10 \log_{10} 2 - 3)}{10^2} = 0 + \frac{3}{10} + \frac{[10(10 \log_{10} 2 - 3)]}{10^2} +$$

$$+ \frac{\{10(10 \log_{10} 2 - 3)\}}{10^2} = 0 + \frac{3}{10} + \frac{0}{10} + \frac{10(10^2 \log_{10} 2 - 30)}{10^3} =$$

$$= 0 + \frac{3}{10} + \frac{0}{10^2} + \frac{1}{10^3} + \frac{10(10^3 \log_{10} 2 - 301)}{10^4}$$

Finaly, $\log_{10} \frac{p}{q}$ can be approximated by the difference of initial k-sections $(\log_{10} p)_k - (\log_{10} q)_k$, yielding a solution (12).

Sine. There is the same ratio between corresponding sides in any two orthogonal triangles with the same angles. It permits us to calculate lengths of sides in a large triangle using a relation of sides in a small triangle if we know one of its sides. In that way we can calculate a distance between two inaccessible places, which was one of the earliest applications of mathematics. It is useful to have those ratios between sides together with the given angles arranged into tables (so it was in the past when there was no need of a great exactness) or to be able to compute them quickly (with a calculator).

A sine of an angle is the ratio between the side opposite this angle and the hypotenuse in an orthogonal triangle. In order to regard a sine as a function $f(x)$ defined for real numbers we have to know how to express the magnitude of an angle as a real number. This is possible if we measure the length of a corresponding arc of a unit circle. We speak about measuring angles in radians. Then we can calculate sin x for $x = \alpha$ by the following algorithm:

To a number α we find an abscissa of a length α and we find an arc of a length α using thread so that we roll up the thread on a (material) circle which has a unit radius. We draw a straight line perpendicular to the diameter of the circle from the end point. The length of the perpendicular will represent sin α (the sign will be — if an end point is located under the diameter of the circle). If we like to improve the exactness of the calculation of sin α, we must enlarge the used unit of length; here we cannot express the error exactly.

Algorithm (3) is not convenient for calculating sin α since we cannot eliminate a sine from inequality (16)

$$a \leqq 10^k \sin \alpha - j < a + 1$$

It is in the university course of Calculus only that suitable algorithms for computing sin x can be found. It is the so-called Taylor's expansion sin x (see e.g. [5, p. 97]),

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \dots$$

or in its finite form

$$\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!} \sin (\Theta_1 x)$$

($0 < \Theta_1 < 1$) where the last member represents and error which we make if we sum up only the terms from $k = 0$ to $k = n$ in that infinite series. Hence we obtain a solution (12) in the form

$$\left|\sin\left(\frac{p}{q}\right) - \frac{P}{Q}\right| \leqq \frac{\left(|\alpha_{k_0}| + \dfrac{1}{10^{k_0}}\right)^{2n+2}}{(2n+2)!}$$

where

$$\frac{P}{Q} = \frac{p}{q} \cdot \frac{1}{1!} - \left(\frac{p}{q}\right)^3 \cdot \frac{1}{3!} + \left(\frac{p}{q}\right)^5 \cdot \frac{1}{5!} + \dots + (-1)^n \cdot \left(\frac{p}{q}\right)^{2n+1} \cdot \frac{1}{(2n+1)!}$$

$$\left|\alpha - \frac{p}{q}\right| \leqq \frac{1}{10^k}, \qquad k_0 \leqq k.$$

We get a solution (13) by using a derivative $(\sin x)' = \cos x$ and from Lagrange's mean-value theorem:

$$\left|\sin \alpha - \sin \frac{p}{q}\right| = |\cos \bar{\alpha}| \left|\alpha - \frac{p}{q}\right| \leqq \left|\alpha - \frac{p}{q}\right|$$

Concluding, we remark that a disadvantage of the algorithms for a calculating $x^{\frac{1}{s}}$, $\log_{10} x$ which are derived using the fundamental algorithm (3) is the fact that the numbers occuring when solving inequality (16) increase very quickly (see a calculation of $\sqrt[3]{2}$, $\log_{10} 2$). In order to increase the exactness of the computation, i.e. of calculating the next terms in a decimal expansion we would have to construct an arithmetics of great numbers.

There are algorithms derived in a university course of mathematics from the Taylor's theorem which converge quickly enough, i.e. we obtain a good exactness of the result using a little number of steps. For instance (see [5, p. 99])

$$\log(1+x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k} + (-1)^{n+2} \frac{x^{n+1}}{(n+1)(1+\Theta_2 x)^{n+1}}$$

where $x > -1$, $0 < \Theta_2 < 1$.

$$(1-x)^{\frac{1}{s}} = \sum_{k=0}^{n} (-1)^k \binom{\frac{1}{s}}{k} x^k + (-1)^{n+1} \binom{\frac{1}{s}}{n+1} x^{n+1} (1-\Theta_3 x)^{\frac{1}{s}-n-1}$$

where $-1 < x < 1$, $0 < \Theta_3 < 1$ and

$$\binom{\frac{1}{s}}{k} = \frac{\frac{1}{s}\cdot\left(\frac{1}{s}-1\right)\ldots\left(\frac{1}{s}-k+1\right)}{k!}$$

We use this formula similarly as for $\sin x$.

Exercises

1. Applying property (14) show that

(i) if $0 < \alpha < \beta$, then $\alpha^{\frac{1}{s}} < \beta^{\frac{1}{s}}$

(ii) if $s_1 < s_2$ and $0 < \alpha < 1$, then $\alpha^{\frac{1}{s_1}} < \alpha^{\frac{1}{s_2}}$

(iii) $\left(\alpha^{\frac{1}{s_1}}\right)^{\frac{1}{s_2}} = \alpha^{\frac{1}{s_1 s_2}}$

(iv) $\left(\frac{p}{q}\right)^{\frac{r}{s}} = \frac{(p^r)^{\frac{1}{s}}}{(q^r)^{\frac{1}{s}}}$

where s, s_1, s_2, r are positive integers. All those properties hold also for positive rational exponents and then they extend also to real exponents. We put $\alpha^{-\gamma} = \frac{1}{\alpha^{\gamma}}$ for a negative exponent.

2. Why does $(-1)^{\sqrt{2}}$ not exist?

3. Using property (17) show that

(i) if $0 < \alpha < \beta$ then $\log_{10} \alpha < \log_{10} \beta$

(ii) $\log_{10} pq = \log_{10} p + \log_{10} q$

(iii) $\log_{10} \frac{p}{q} = \log_{10} p - \log_{10} q$

(iv) $\log_{10} \alpha^{\frac{r}{s}} = \frac{r}{s} \log_{10} \alpha$

(v) $\log_{\alpha} \beta \cdot \log_{\beta} \alpha = 1$.

All these properties of logarithm extend also to real numbers.

4. Find $\alpha^{\frac{1}{2}}$ with an exactness of $\frac{1}{10^3}$ at the point $\alpha = \sqrt[3]{2}$ using a decimal expansion of $\sqrt[3]{2}$, i.e. find a rational number $\frac{P}{Q}$ so that

$$\left|(\sqrt[3]{2})^{\frac{1}{2}} - \frac{P}{Q}\right| \leqq \frac{1}{10^3}.$$

How many digits of $\frac{P}{Q}$ will be equal to digits of $(\sqrt[3]{2})^{\frac{1}{2}}$?

5. How many terms from a decimal expansion of $\sqrt{2}$ are needed to find a rational number $\frac{P}{Q}$ such that

$$\left|\sqrt{2}^{\sqrt{2}} - \frac{P}{Q}\right| \leqq \frac{1}{10^{10}}?$$

6. Find $\log_{10} 2$ with an exactness 10^{-5} by using the Taylor's theorem.

7. Express $\cos x$, $\operatorname{tg} x$ in terms of $\sin x$.

8. Review the addition formulas for trigonometrical functions.

9. Calculate $\sin \sqrt{2}$ with an exactness 10^{-5}. How many exact digits of $\sin \sqrt{2}$ shall we find by it?

10. Calculate $\sin \sqrt{2}$ using a tread and a unit circle. Investigate experimentaly how the exactness improves with increasing unit length.

REFERENCES

[1] Nyulassyová, E.: Constructions of the field of real numbers and methods of their education at the secondary school, The second degree thesis, Bratislava, 1981 (in Slovak).

[2] Nečajev, V. I.: Numerical systems, Prosveščenie, Moskva, 1975 (in Rusian).

[3] Smida, J.—Šedivý, J.—Lukátšová, J.—Vocelka, J.: Mathematics for the first class of a secondary school, SPN, Bratislava, 1984 (in Slovak).

[4] Odvárko, O.—Fořt, J.—Novák, B.: Mathematics for a secondary school, SPN, Bratislava, 1978 (in Slovak).

[5] Kluvánek, I.—Mišík, L.—Švec, M.: Mathematics II., SVTL, Bratislava, 1961 (in Slovak).

[6] Franek, M.: Über eine Konstruktion reelen Zahlen I., A.F.R.N.U.C., XXVI, 1972, 57—81.

[7] Nikoľskij, S. M.: Elements of mathematical analysis, Moskva, 1981 (in Russian).

Author's address:
Eva Nyulassyová
Katedra geometrie MFF UK
Mlynská dolina
Bratislava
842 15

Received: 5. 1. 1985

Oto Strauch
Matematický ústav SAV
Obrancov mieru 49
Bratislava
814 73

SÚHRN

REÁLNE ČÍSLA NA STREDNEJ A VYSOKEJ ŠKOLE, I.

Eva Nyulassyová — Oto Strauch, Bratislava

V práci je podaný systematický výklad reálnych čísel a funkcií s použitím dekadických rozvojov.

РЕЗЮМЕ

ДЕЙСТВИТЕЛЬНЫЕ ЧИСЛА В СРЕДНЕМ И ВЫСШЕМ УЧЕБНОМ ЗАВЕДЕНИИ, I

Ева Нюлашиова — Ото Штраух, Братислава

В работе систематически излагается теория действительных чисел и действительных функций на основе декадического представления действительного числа.