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ON AN ALTERNATIVE AXIOM SYSTEM FOR NONSTANDARD CALCULUS

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In the history of differential and integral calculus there were two methods: the method of limits and the method of infinitesimals. Of course, the second method has been based on some rigorous foundations only a few years ago ([7], see also [1], [8], [2]). Therefore many methodical problems have not been solved definitively. Especially for teaching is not very convenient neither using a fine apparatus of mathematical logic, nor the ultrapower construction. So various authors suggested ([5], [6], [3], [4]) certain axiomatic approaches. In the paper we present a system of axioms. As an illustration of the power of our system we present: 1) the equivalence of a standard and a nonstandard notion (theorems 1, 3, 4, 5); 2) the Cantor theorem (Theorem 2) as a classical assertion proved in the new axiomatic system; 3) a nonstandard analysis theorem (Theorem 6). All these assertions present basic facts for the calculus.

Axioms

1. An ordered field H and its subfield $R \subset H$ are given.

(The elements of R are caled real numbers, the elements of H hyperreal numbers. We shall write $x \approx y$ iff $|x - y| < r$ for every positive real r. An element $x \in H$ is called infinitesimal, if $x \approx 0$.)

2. There is a mapping $*\colon 2^R \to 2^H$ such that $A \subset {}^*A$ for all $A \subset R$; ${}^*R = H$. Further, if N is the set of all positive integers (i.e. N is the minimal set containing 1 and closed under addition), then ${}^*N \backslash N \neq \emptyset$ and $x > n$ for all $x \in {}^*N \backslash N$ and $n \in N$.

(The axiom implies that there is a non-zero infinitesimal. Namely, if $x \in {}^*N \backslash N$, then $0 < \frac{1}{x} < \frac{1}{n}$ for all $n \in N$, so $\frac{1}{x} \approx 0$, $\frac{1}{x} \neq 0$.)

3. For every interval $\langle a, b\rangle \subset R$ and every $x \in {}^*\langle a, b\rangle$ there exists exactly one $r \in \langle a, b\rangle$ such that $x \approx r$. (Denote this element r by st(x).) If $x \in {}^*\langle a, b\rangle$, then $a \leqq x \leqq b$. If $x \leqq y$, $x, y \in {}^*\langle a, b\rangle$, then $\mathrm{st}(x) \leqq \mathrm{st}(y)$.

4. For every $f\colon A \times B \to C$, $A, B, C \subset R$ there is an extension

$$^*f\colon {}^*A \times {}^*B \to {}^*C$$

of f such that the following properties are satisfied:

$$^*(f+g) = {}^*f + {}^*g, \quad {}^*(f\cdot g) = {}^*f\cdot{}^*g$$

and the image of any constant function is a constant function. If $f\colon X \to Y$, $g\colon Y \to Z$, then $^*(g \circ f) = {}^*g \circ {}^*f$.

5. For any $f\colon \langle c, d\rangle \times N \to R$ the following two statements are equivalent:

(i) There is $n_0 \in N$ such that for every $n \in \langle c, d\rangle$ it is $f(x, n) \geqq 0$.

(ii) $^*f(x, n) \geqq 0$ for every $x \in {}^*\langle c, d\rangle$ and every $n \in {}^*N \setminus N$.

The last axiom presents a non-standard characterization of the uniform convergence of a sequence of functions. Possible justification: if f is nonnegative from a moment, then it can be hoped that f will be nonnegative in a long future, too (see [9]). And contrary. Of course, sometimes we shall need only some special cases of the axiom:

5′. For any sequence $f\colon N \to R$ the following two statements are equivalent:

(i) There is $n_0 \in N$ such that $f(n) \geqq 0$ for every $n \in N$, $n \geqq n_0$.

(ii) $^*f(n) \geqq 0$ for every $n \in {}^*N \setminus N$.

5″. For every $f\colon \langle c, d\rangle \to R$ the following two statements are equivalent:

(i) $f(x) \geqq 0$ for every $x \in \langle c, d\rangle$.

(ii) $^*f(x) \geqq 0$ for every $x \in {}^*\langle c, d\rangle$.

It is quite easy (starting with R as a complete ordered field) to construct a field H and correspondences $A \mapsto {}^*A$, $f \mapsto {}^*f$ such that all axioms 1—5 are satisfied (see [4]). We fix an ultrafilter F of subsets of N (i.e. F is closed under intersections and supersets, does not contain $\emptyset$ and any finite set and for every $E \subset N$ it contains either E or $N \setminus E$). If $x, y \in R^N$ are two sequences of real numbers, then $x \sim y$ iff $\{n \in N;\ x_n = y_n\} \in F$. H is the factor space $R^N/\!\sim$, so $H = \{[(x_n)_n];\ x_n \in R\}$, where $[(x_n)_n] = \{(y_n)_n;\ (x_n)_n \sim (y_n)_n\}$. The set H becomes an ordered field if one defines

$$[(x_n)_n] + [(y_n)_n] = [(x_n + y_n)_n],$$

$[(x_n)_n] < [(y_n)_n]$ iff $\{n;\ x_n < y_n\} \in F$ etc. For $A \subset R$ we define

$$^*A = \{[(x_n)_n] \in H;\ x_n \in A\}.$$

Finally for $f\colon A \times B \to C$ we define $^*f\colon {}^*A \times {}^*B \to {}^*C$ by the formula

$$^*f([(x_n)_n], [(y_n)_n]) = [(f(x_n, y_n))_n].$$

Of course, these definitions are convenient for graduated mathematicians, but not so for students. Therefore we recommend a simple axiomatic system rather than this construction.

Sequences

Of course, we assume that the reader is aquainted the standard theory. The aim of this lines is only to show that our axioms are sufficiently strong for constructing a good theory. So it should be possible to built tho whole theory by the help of nonstandard methods only, too.

Theorem 1. Let $(f(n))_n$ be a sequence of real numbers, $a \in R$. Then $\lim_{n \to \infty} f(n) = a$ if and only if $*f(n) \approx a$ for every $n \in *N \setminus n$.

Proof. By the standard definition $\lim_{n \to \infty} f(n) = a$ if and only if

$$\forall \varepsilon > 0 \, \exists n_0 \in N \, \forall n \geqq n_0 \colon |f(n) - a| \leqq \varepsilon.$$

By the axioms 5′ and 4 the following statements are equivalent (under fixed $\varepsilon > 0$):

$$\exists n_0 \mathsf{A} n \geqq n_0 \colon f(n) - a + \varepsilon \geqq 0 \wedge \varepsilon + a - f(n) \geqq 0$$

$$\forall n \in *N \setminus N \colon *f(n) - a + \varepsilon \geqq 0 \wedge \varepsilon + a - *f(n) \geqq 0$$

$$\forall n \in *N \setminus N \colon |*f(n) - a| \leqq \varepsilon$$

Since $|*f(n) - a| \leqq \varepsilon \, (n \in *N \setminus N)$ for every $\varepsilon > 0$, we have that $*f(n) \approx a$ for every $n \in *N \setminus N$.

Theorem 2. The axioms 1—4 and 5′ imply the Cantor theorem: If

$$a_n \leqq a_{n+1} \leqq b_{n+1} \leqq b_n \qquad (n = 1, 2, \ldots)$$

are real numbers and $\lim_{n \to \infty} (b_n - a_n) = 0$, then there is exactly one

$$c \in \bigcap_{n=1} \langle a_n, b_n \rangle.$$

Proof. Evidently $a_n \leqq b_m$ for all $n, m \in N$, so by the axioms 5′ and 4 $a_n \leqq$ $\leqq *b(k)$ for every $k \in *N \setminus N$ and $n \in N$. Therefore $a_n \leqq \mathrm{st}(*b(k))$. Evidently $b_m =$ $= a_m + (b_m - a_m)$, so $*b(k) = *a(k) + *(b - a)(k) \approx *a(k) + 0 = *a(k)$. Therefore $a_n \leqq \mathrm{st}(*b(k)) = \mathrm{st}(*a(k))$. Hence 5′ and 4 give once more $*a(p) \leqq \mathrm{st}(*a(k))$ for all $k, p \in *N \setminus N$. By 3 we obtain $\mathrm{st}(*a(p)) \leqq \mathrm{st}(*a(k))$ for all $k, p \in *N \setminus N$, so that $\mathrm{st}(*a(p)) = \mathrm{st}(*a(k))$. Denote this common value by c, hence $*a(k) \approx c$ for all $k \in *N \setminus N$, so that $\lim_{n \to \infty} a_n = c$ by Theorem 1. Since $(a_n)_n$ is non-decreasing, we have $a_n \leqq c$ $(n = 1, 2, \ldots)$. On the other hand, the inequalities $a_m \leqq b_n$ $(m = 1,$

2, ...). Imply $c = \lim_{m \to \infty} a_m \leqq b_n$, so that $a_n \leqq c \leqq b_n$ for every $n \in N$.

Continuous functions

Theorem 3. Let f be a function whose domain contains a neighbourhood of a point x_0. Then f is continuous at x_0 if and only if the following implication hold:

$$x \approx x_0 \Rightarrow {}^*f(x) \approx f(x_0)$$

Proof. $\Rightarrow$

Let $x \approx x_0$. Since f is continuous at x_0, the following holds:

$$\forall \varepsilon > 0 \, \exists \, \delta > 0 \, \forall r \in R \colon |r - x_0| < \delta \Rightarrow |f(r) - f(x_0)| < \varepsilon.$$

Put

$$\langle c, d \rangle = \left\langle x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2} \right\rangle$$

in the axiom 5″. By preceding

$$f(x_0) + \varepsilon - f(r) > 0 \wedge f(r) - f(x_0) + \varepsilon > 0$$

for every $r \in \langle c, d \rangle$. Hence by the axiom 5″

$$f(x_0) + \varepsilon - {}^*f(x) \geqq 0 \wedge {}^*f(x) - f(x_0) + \varepsilon \geqq 0$$

so that

$$|{}^*f(x) - f(x_0)| \leqq \varepsilon.$$

Since the inequality holds for every $\varepsilon > 0$, we obtain ${}^*f(x) \approx f(x_0)$.

$\Leftarrow$

Let $a_n \to x_0$. Then ${}^*a(n) \approx x_0$ for every $n \in {}^*N \backslash N$ (Theorem 1). Therefore ${}^*f({}^*a(n)) \approx f(x_0)$, hence by Theorem 1

$$f(x_0) = \lim_{n \to \infty} f \circ a(n) = \lim_{n \to \infty} f(a_n)$$

so that f is continuous at x_0.

Theorem 4. f is uniformly continuous on $\langle a, b \rangle$ if and only if the following implication holds:

$$x, y \in {}^*\langle a, b \rangle, \; x \approx y \Rightarrow {}^*f(x) \approx {}^*f(y)$$

Proof. $\Rightarrow$

Let $x, y \in {}^*\langle a, b \rangle$, $x \approx y$. Since f is uniformly continuous on $\langle a, b \rangle$ the following statement holds:

$$\forall \varepsilon > 0 \, \exists \, \delta > 0 \, \forall r, s \in \langle a, b \rangle \colon |r - s| < \delta \Rightarrow |f(r) - f(s)| < \varepsilon$$

Put in 5″ $\langle c, d\rangle = \left\langle \mathrm{st}(x) - \frac{\delta}{2}, \mathrm{st}(x) + \frac{\delta}{2}\right\rangle$ $\left(\text{resp. } \langle c, d\rangle = \left\langle a, a + \frac{\delta}{2}\right\rangle \text{ if } x = a,\right.$ $\left.\langle c, d\rangle = \left\langle b - \frac{\delta}{2}, b\right\rangle, \text{ if } x = b\right)$. Then for every $r \in \langle c, d\rangle$

$$f(r) - f(\mathrm{st}(x)) + \varepsilon > 0 \wedge f(\mathrm{st}(x)) - f(n) + \varepsilon > 0$$

hence by the axiom 5″

$$*f(y) - f(\mathrm{st}\, x) + \varepsilon \geqq 0 \wedge f(\mathrm{st}\, x) - *f(y) + \varepsilon \geqq 0$$

so that

$$|*f(y) - f(\mathrm{st}\, x)| \leqq \varepsilon.$$

Since the last inequality holds for every $\varepsilon > 0$, we have $*f(y) \approx f(\mathrm{st}\, x)$. Since f is continuous at $\mathrm{st}(x)$ (right or left continuous if $\mathrm{st}(x) = a$ or $\mathrm{st}(x) = b$) and $\mathrm{st}(x) \approx x$, we have by Theorem 3 $f(\mathrm{st}(x)) \approx *f(x)$, so that $*f(x) \approx *f(y)$.

$\Leftarrow$

Let f not be uniformly continuous on $\langle a, b\rangle$. Then the following statement holds:

$$\exists \varepsilon > 0 \,\forall n \in N \,\mathrm{E}\, x_n, y_n \in \langle a, b\rangle : |x_n - y_n| < \frac{1}{n} \wedge |f(x_n) - f(y_n)| \geqq \varepsilon.$$

Then $|*x(n) - *y(n)| \leqq \frac{1}{n}$ for every $n \in *N \backslash N$. Therefore $*x(n) \approx *y(n)$ for every $n \in *N \backslash N$. So by the assumption $*f(*x(n)) \approx *f(*y(n))$ for every $n \in *N \backslash N$. On the other hand, by the axiom 5′ we obtain $|*f(*x(n)) - *f(*y(n))| \geqq \varepsilon$ for every $n \in *N \backslash N$, which is a contradiction.

Now we are able to repeat one of the most beautiful nonstandard proofs: the proof of the uniform continuity of every function f continuous on a compact interval $\langle a, b\rangle$. Namely, if $x \approx y$, $x, y \in *\langle a, b\rangle$, then there are (by the axiom 3) $r, s \in \langle a, b\rangle$ such that $x \approx r$, $y \approx s$, hence $r \approx s$, so that $r = s$. The continuity of f at r and Theorem 3 imply that $*f(x) \approx f(r) \approx *f(y)$. Therefore f is uniformly continuous by Theorem 4.

Integral

We suppose for this moment that we have constructed the standard theory of the definite integral of continuous functions. So for every function b continuous on $\langle c, d\rangle$

$$\int_c^d b(x)dx = \lim_{n \to \infty} g(n),$$

where

$$g(n) = \sum_{i=0}^{n-1} b\left(c + \frac{d-c}{n} i\right) \frac{d-c}{n}.$$

Theorem 5. $\int_c^d b(x)dx = \text{st}(*g(n))$ for every $n \in *N \setminus N$.

Proof. The relation $\int_c^d b(x)dx = \lim_{n \to \infty} g(n)$ implies (by Theorem 1)

$$\int_c^d b(x)dx \approx *g(n)$$

for every $n \in *N \setminus N$.

The preceding theorem has rather philosphical meaning. Namely, if $n \in *N \setminus N$, then $dx = \frac{d-c}{n} \approx 0$, so $*g(n)$ can be interpreted as a sum of n (i.e. infinitely many) infinitely thin rectangles with volumes $b(x)dx$. But a practical meaning for applications has the following infinite sum theorem.

Theorem 6. Let b be a continuous function on a compact interval $\langle c, d \rangle$. Let L be an additive function of an interval (i.e.

$$L(\langle c, d \rangle) = \sum_{i=0}^{n-1} L(\langle x_i, x_{i+1} \rangle),$$

if $c = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = d$). Put (for $n \in N$ and $x \in \langle a, b \rangle$)

$$f(n, x) = \frac{L\left(\left\langle x, x + \frac{d-c}{n} \right\rangle\right)}{\frac{d-c}{n}}$$

and assume that $*f(n, x) \approx *b(x)$ for every $n \in *N \setminus N$, $x \in *\langle c, d \rangle$. Then

$$L(\langle c, d \rangle) = \int_c^d b(x)dx.$$

Proof. By the axiom 5 it is easy to see that $\lim_{n \to \infty} f(n, x) = b(x)$ uniformly on $\langle c, d \rangle$. So the following statement is satisfied:

$$\forall \varepsilon > 0 \exists n_0 \forall n \geqq n_0 \forall x \in \langle c, d \rangle : \left| L\left(\left\langle x, x + \frac{d-c}{n} \right\rangle\right) - b(x) \frac{d-c}{n} \right| < \frac{\varepsilon}{n}.$$

Put $x_i = c + \frac{d-c}{n} i$ $(i = 0, 1, \ldots, n-1)$ instead of x and sum these inequalities.

Then (by the additivity of L)

$$\left|L(\langle c, d\rangle) - \sum_{i=0}^{n-1} b(x_i) \frac{d-c}{n}\right| = |L(\langle c, d\rangle) - g(n)| < \varepsilon,$$

hence by the axiom 5′

$$|L(\langle c, d\rangle) - {}^*g(n)| \leqq \varepsilon$$

for every $n \in {}^*N \setminus N$. Therefore $L(\langle c, d\rangle) \approx {}^*g(n)$, hence

$$L(\langle c, d\rangle) = \int_c^d b(x)dx$$

by Theorem 5.

In applications usually one writes $dx = \frac{d-c}{n}$ for $n \in {}^*N \setminus N$, so dx is an infinitesimal. Then the third assumption of Theorem 6 can be rewritten in the form $\frac{L(\langle x, x + dx\rangle)}{dx} \approx b(x)$, so $b(x)$ is something like the density of the function L at the point x. The application is the following. Let $L(\langle c, d\rangle)$ be e.g. the volume of the rotation solid generated by a positive continuous function h over $\langle c, d\rangle$. For infinitely thin set (i.e. the set over $\langle x, x + dx\rangle$) we obtain a cylinder with the base radius $h(x)$ and the thickness dx, thus with the volume

$$\pi h(x)^2 dx = L(\langle x, x + dx\rangle).$$

Therefore

$$\frac{L(\langle x, x + dx\rangle)}{dx} = \pi h(x)^2 = b(x).$$

Theorem 6 now implies the equality

$$L(\langle c, d\rangle) = \int_c^d b(x)dx = \pi \int_c^d h(x)^2 dx.$$

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SÚHRN

O ALTERNATÍVNOM AXIOMATICKOM SYSTÉME PRE NEŠTANDARDNÝ DIFERENCIÁLNY A INTEGRÁLNY POČET

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Pomocou neveľkého počtu jednoduchých axióm pre hyperreálne čísla sa dokazujú neštandardné charakterizácie limity postupnosti, spojitosti a rovnomernej spojitosti, Cantorova veta o vpísaných intervaloch a veta o súčte nekonečne veľa nekonečne malých veličín.

РЕЗЮМЕ

ОБ АЛЬТЕРНАТИВНОЙ АКСИОМАТИЧЕСКОЙ СИСТЕМЕ ДЛЯ НЕСТАНДАРДНОГО ДИФФЕРЕНЦИАЛЬНОГО И ИНТЕГРАЛЬНОГО ИСЧИСЛЕНИЯ

Белослав Риечан, Братислава

При помощи небольшого множества простых аксиом для гипердействительных чисел доказываются нестандардные характеризации предела последовательности, непрерывности и равномерной непрерывности, теорема Кантора о включенных сегментох и теорема о сумме бесконечного количества бесконечно малых величин.