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AN INTERPOLATION FORMULA AND SOME COMBINATORIAL IDENTITIES

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Let f be a real function on the interval $[a, +\infty)$. If $h > 0$, let $a_k = a + kh$ for $k = 0, 1, \ldots$. Denote by $f(a_0, \ldots, a_n)$ the divided difference of order n of the function f at $a_0, \ldots, a_n$ and by P_n the interpolation polynomial of f on the nodes $a_0, \ldots, a_n$ in the Newton form, i.e.

$$P_n(x) = f(a_0) + (x - a_0)f(a_0, a_1) + \ldots + (x - a_0) \ldots (x - a_{n-1})f(a_0, \ldots, a_n)$$

Writing

$$\omega_{k+1}(x) = (x - a_0) \ldots (x - a_k)$$

we have

$$f(x) = P_n(x) + R_n(x)$$

where

$$R_n(x) = \omega_{n+1}(x) \sum_{k=0}^{n} \frac{f(a_k) - f(x)}{\omega_{n+1}(a_k)(a_k - x)}$$

If the derivative f' of f on $[a, +\infty)$ exists, we have

$$f'(x) = P_n'(x) + R_n'(x)$$

with

$$R_n'(x) = \omega_{n+1}'(x) \sum_{k=0}^{n} \frac{f(a_k) - f(x)}{\omega_{n+1}(a_k)(a_k - x)} + \omega(x) \sum_{k=0}^{n} \frac{f(a_k) - f(x) - (a_k - x)f'(x)}{\omega_{n+1}'(a_k)(a_k - x)^2} \tag{1}$$

Suppose f has continuous derivatives up to the order $n + 1$; then (cf. for instance [3], th. 4.1)

$$R_n'(x) = \frac{(-1)^n h^n}{n+1} f^{(n+1)}(\xi) \tag{2}$$

the number ξ laying between the least and the greatest of the numbers x, a_0, $\ldots$, a_n.

Since, in our case,

$$\omega'_{k+1}(a) = (-1)^k k!\, h^k$$

$$f(a_0, \ldots, a_k) = \frac{1}{h^k k!} \sum_{j=0}^{k} (-1)^k \binom{k}{j} f(a_j)$$

we get, after simple calculations,

$$P'_n(a) = \sum_{k=0}^{n-1} \omega'_{k+1}(a) f(a_0, \ldots, a_{k+1}) =$$

$$= \frac{1}{h}\left(\sum_{k=1}^{n} (-1)^{k+1} f(a_k) \sum_{j=k}^{n} \frac{1}{j}\binom{j}{k} - f(a)\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\right) =$$

$$= \frac{1}{h}\left(\sum_{k=1}^{n} (-1)^{k+1} \frac{f(a_k)}{k} \sum_{j=k}^{n} \binom{j-1}{k-1} - f(a)\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\right) =$$

$$= \frac{1}{h}\left(\sum_{k=1}^{n} (-1)^{k+1} \frac{f(a_k)}{k} \binom{n}{k} - f(a)\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\right)$$

So we have

$$f'(a) = \frac{1}{h}\left(\sum_{k=1}^{n} (-1)^{k+1} \frac{f(a_k}{k} \binom{n}{k} - f(a)\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\right) + R'_n(a) \tag{3}$$

Take $a = 0$, $h = 1$. If $f = 1$, we obtain by (2)

$$\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \ldots + \frac{(-1)^{n-1}}{n}\binom{n}{n} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \tag{4}$$

Similarly, for $f(x) = x^{m+1}$, $0 < m < n$, it follows that

$$\binom{n}{1} - 2^m\binom{n}{2} + \ldots + (-1)^{n-1} n^m \binom{n}{n} = 0 \tag{5}$$

and for $f(x) = x$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots = 0 \tag{6}$$

If $f(x) = x^{n+1}$, then (2) and (3) yield

$$1^n\binom{n}{1} - 2^n\binom{n}{2} + \ldots + (-1)^{n-1} n^n \binom{n}{n} = (-1)^{n-1} n!$$

i.e.

$$(7) \qquad \binom{n}{0}n^n - \binom{n}{1}(n-1)^n + \dots + (-1)^{n-1}\binom{n}{n-1} = n!$$

More generally, let $m \geqslant 0$ be an integer and $f(x) = x^{m+1}$; then

$$(8) \qquad |f'(0) - P_n'(0)| = |R_n'(0)| =$$
$$= \binom{n}{0}n^m - \binom{n}{1}(n-1)^m + \dots + (-1)^{n-1}\binom{n}{n-1}1^m + (-1)^n\binom{n}{n}0^m$$

is the number of all mappings of an m-set onto any n-set (see, for example [1], chap. 3, p. 120, [2], p. 18—19, [3], p. 129—130). So, the formulae (5), (6), (7) are particular cases of (8) (if we put $0^0 = 1$).

Similarly, further identities can be obtained by special choice of the function f.

We mention finally the formulae

$$(9) \qquad \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\binom{n}{k}\sin k = 1 + O\left(\frac{1}{n}\right)$$

$$(10) \qquad \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\binom{n}{k}\cos k = 1 + \frac{1}{2} + \dots + \frac{1}{n} + O\left(\frac{1}{n}\right)$$

which follow from (2) and (3).

For other methods of proving combinatorial identities see e.g. J. Kaucký: Kombinatorické identity, Bratislava, 1975, or J. Riordan: Combinatorial identities, New York, 1968.

BIBLIOGRAPHY

[1] Bourbaki, N.: Théorie des ensembles. Paris, Hermann 1963.
[2] Kosmák, A.: Kombinatorická teória pravdepodobnosti. Bratislava, Alfa 1979.
[3] Ralston, A.: A first course in numerical analysis. McGraw-Hill 1965.
[4] Sačkov, V. N.: Kombinaronyje metody diskretnoj matematiki. Moscow 1977.

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SÚHRN

NUMERICKÉ DERIVOVANIE A KOMBINATORICKÉ IDENTITY

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Dokazuje sa, že kombinatorické vzťahy (4)—(10) vyplývajú z formuly (3) vhodnou voľbou funkcie f.

РЕЗЮМЕ

ОБ ОДНОЙ ИНТЕРПОЛЯЦИОННОЙ ФОРМУЛЕ И НЕКОТОРЫХ КОМБИНАТОРНЫХ СООТНОШЕНИЯХ

Ладислав Космак

Показывается, что соотношения (4)—(10) непосредственно вытекают из формулы (3).

DIDAKTIKA
MATEMATIKY