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RELATION BETWEEN CHOMSKY HIERARCHY AND COMMUNICATION COMPLEXITY HIERARCHY

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Abstract

We study communication complexity from the point of view of the Language theory. We establish the relation between Chomsky hierarchy and the hierarchy of the language families determined by communication complexity.

Introduction

We shall study a new complexity measure in Language theory which, informally, can be defined in the following way. Suppose a language $L \subseteq (\{0, 1\}^2)^*$ is to be recognized by two distant computers. Each computer receives half of the input bits, and the computation proceeds using some protocol for communication between these two computers. The minimal number of bits that must be exchanged in order to successfully recognize $L \subseteq \{0, 1\}^{2n}$, minimized over all partitions of the input bits into two equal parts, and considered as a function of n, is called the communication complexity of L. The communication complexity defined in such a way provides a direct lower bound for minimal bisection width [1] of any chip recognizing L which was the main reason to introduce it.

The hierarchy of language families determined by communication complexity was first studied in [5], where it was shown that most languages cannot be recognized in $n - 1$ communication complexity, and that, for $0 \leqslant f(n) \leqslant \log_2 n$, $f(n) + 1$ communication complexity is more powerful than $f(n)$ communication complexity. But the proofs of these results were insufficient what is pointed out in [2]. In [3] it is shown that a substantionally larger number of languages can be recognized within communication complexity $f(n)$ than within communication complexity $cf(n)$, for a $c < 1$. Closely related hierarchy results according to special types of communication complexity were obtained in [3, 4, 5].

We shall deal with the general model of communication complexity from the standpoint of Language theory. The results concerning the relation between Chomsky hierarchy and communication complexity hierarchy will be obtained.

This paper is divided in two sections. In Section 1 the basic definitions are given. We shall establish the relation between the Chomsky hierarchy and the communication complexity hierarchy in Section 2.

1. Definitions

Now, let us formally define the model of communication complexity in the same way as in [5].

A *protocol on* $2n$ *inputs* is a pair $D_n = (\Pi, \Phi)$, where

1. Π is a partition of $\{1, 2, \dots, 2n\}$ into two equal sets S_{I} and S_{II}. This corresponds to the partition of the input into the two halves for the two computers.

2. Φ is a function from $\{0, 1\}^n \times \{0, 1, \$\}^*$ to $\{0, 1\}^* \cup \{\text{accept, reject}\}$. Intuitively, the first argument of Φ is the local part of the input, while the second argument involves all previous messages, with $ serving as the delimiter between messages. The result of Φ is the next message. For a given string c in $\{0, 1, \$\}^*$, the function Φ has the property that for no two y, y' in $\{0, 1\}^n$ is the case that $\Phi(y, c)$ is a proper prefix of $\Phi(y', c)$. This *prefix-freeness* property assures that the exchanged messages are self-delimiting, and that no extra "end of transmission" symbol is required.

The *computation* of D_n on an input word x in $\{0, 1\}^{2n}$ is the string $c = c_1 \$ c_2 \$ \dots \$ c_k \$ c_{k+1}$, where $k \geqslant 0$, $c_1, \dots, c_k \in \{0, 1\}^*$, $c_{k+1} \in \{\text{accept, reject}\}$, such that for each integer j, $0 \leqslant j \leqslant k$, we have

(1) if j is even, then $c_{j+1} = \Phi(x_{\mathrm{I}}, c_1 \$ c_2 \$ \dots \$ c_j)$, where x_{I} is the input x restricted to the set S_{I} and

(2) if j is odd, then $c_{j+1} = \Phi(x_{\mathrm{II}}, c_1 \$ c_2 \$ \dots \$ c_j)$, where x_{II} is the input x restricted to the set S_{II}.

Let $L \subseteq \{0, 1\}^*$ be a language and $\Delta = \langle D_n \rangle$ be a sequence of deterministic protocols. We say Δ *recognizes* L if, for each n and each x in $\{0, 1\}^{2n}$, the computation of D_n on input x is finite, and ends with accept iff $x \in L$. Let f be a function from integers to integers. We say that L *is recognizable within communication* f, $L \in COMM(f)$, if there is a sequence of protocols $\Delta = \langle D_n \rangle$ such that for all n and each x in $\{0, 1\}^{2n}$ the computation of D_n on x is of the length at most $f(n)$.

Now, we shall formulate an assertion which will show the power of the partition in protocols and which will be a nice example of the computation of a protocol. Before we give this result we call attention to the fact that every language is recognizable within communication n, where $2n$ is the length of the input word.

Lemma 1. Let Π be the partition of $\{1, 2, \ldots, 2n\}$ into sets $S_I = \{1, 2, \ldots, n\}$ and $S_{II} = \{n+1, n+2, \ldots, 2n\}$ for all n. Then there exists a language L fulfilling the following conditions:

(1) $L \in COMM(1)$,

(2) For all n and all Φ, the protocols $D_n = (\Pi, \Phi)$ cannot recognize $L \cap \{0, 1\}^{2n}$ within communication $n - 1$.

Proof. Let us consider the language $L = \{ww \mid w \in \{0, 1\}^*\}$. We shall show first that $L \in COMM(1)$. For each n we construct the protocol $D_n = (\Pi', \Phi')$, where Π' is the partition of $\{1, 2, \ldots, 2n\}$ into sets

$$P_I = \{1, 2, \ldots, [n/2], n+1, n+2, \ldots, n+[n/2]\}$$

and

$$P_{II} = \{[n/2]+1, [n/2]+2, \ldots, n, n+[n/2]+1, n+[n/2]+2, \ldots, 2n\},$$

and Φ' is described as follows. Let $y = a_1 \ldots a_{2n}$, $a_i \in \{0, 1\}$ be an input word. Then, for all x in $\{0, 1\}^n$, z in $\{0, 1\}$,

$\Phi'(x) = reject$ iff there exists j in $\{1, 2, \ldots, [n/2]\}$ such that $a_j \neq a_{n+j}$,

$\Phi'(x) = a_{n+[n/2]}$ (1) if for all j in $\{1, 2, \ldots, [n/2]\}$ follows $a_j = a_{n+j}$ and n is odd (even),

$\Phi'(x, z) = reject$ iff there exists j in $\{[n/2]+1, \ldots, n\}$ such that $a_j \neq a_{n+j}$,

$\Phi'(x, z) =$ accept iff for all j in $\{[n/2]+1, \ldots, n\}$ follows $a_j = a_{n+j}$.

Clearly, for all n, D_n accepts $L \cap \{0, 1\}^{2n}$.

Now, we shall show that $L \cap \{0, 1\}^{2n}$ can be accepted by no $D_n = (\Pi, \Phi)$ using at most $n - 1$ communication bits, where Π is the partition of $\{1, 2, \ldots, 2n\}$ into sets $S_I = \{1, 2, \ldots, n\}$ and $S_{II} = \{n+1, n+2, \ldots, 2n\}$. We prove it by contradiction. Let $A_n = (\Pi, \Phi)$ be a protocol accepting L within communication $n - 1$.

Now, we shall introduce a fact which we shall use to prove that if A_n accepts all words in $L \cap \{0, 1\}^{2n}$, then it has to accept a word not in L.

Let z be an input word in $\{0, 1\}^{2n}$ and let z_I (z_{II}) be the part of z restricted to the computer I (II) according to Π_0 which is an arbitrary partition of $\{1, 2, \ldots, 2n\}$. Then we shall write $z = \Pi_0^{-1}(z_I, z_{II})$. (Clearly, for Π here considered, $\Pi^{-1}(z_I, z_{II}) = z_I z_{II}$.) Let there exist an accepting computation $d = c_1 \$ c_2 \$ \ldots \$ c_k$ of a $D_n = (\Pi_1, \Phi_1)$ for two different input words x, y in $\{0, 1\}^{2n}$. Then D_n must accept the input words $v = \Pi_1^{-1}(x_I, y_{II})$ and $u = \Pi_1^{-1}(y_I, x_{II})$, where x_I, x_{II} (y_I, y_{II}) are restrictions of x (y) according to Π_1. It follows from the fact that $\Phi_1(x_I) = \Phi_1(y_I) = c_1$ is the first step of D_n computation on all input words x, y, v, u, and $\Phi_1(x_{II}, c_1) = \Phi_1(y_{II}, c_1) = c_2$ is the second step of D_n computation on x, y, v, u and so on the arguments of Φ_1 are such that all steps of D_n computation on x, y, v, u are the same.

Let us consider all accepting computations of A_n on the 2^n words belonging to $L \cap \{0, 1\}^{2n}$. Since the number of all A_n accepting computations is bounded by 2^{n-1}, there exist two different words w_1, w_2 in $\{0, 1\}^n$ such that the input words w_1w_1 and w_2w_2 have the same accepting computation. Realizing the fact introduced above we obtain that $\Pi^{-1}(w_1, w_2) = w_1w_2$ and $\Pi^{-1}(w_2, w_1)$ are accepted by A_n. But the words w_1w_2 and w_2w_1, for $w_1 \neq w_2$, do not belong to L, which is a contradiction.

We note that Lemma 1 can be simply generalized for any particular partition Π.

To conclude this section we give some notation used in what follows. Let i be a natural number. Then $BIN_j(i)$ is the binary code of i on j bits (for example, $BIN_6(5) = 000101$). We shall denote, for a word w, the number of symbols b in w by $\# b(w)$. Let m be a real number. Then $[m]$ ($\{m\}$) is the ceiling (floor) of m.

2. Chomsky hierarchy and communication complexity

We begin to study the relation between the Chomsky hierarchy and the communication complexity hierarchy with the simplest families of languages. We consider the family of regular languages — $\mathscr{R}$ on one side, and the language families $COMM(c)$, where c is a constant, on the other side.

Theorem 1. For all L in $\mathscr{R}$ there exists a constant c such that $L \in COMM(c)$.

Proof. Let L be in $\mathscr{R}$ which means that there exists a deterministic finite automaton A recognizing L. Let A have s states $p_1, \ldots, p_s$. We show that L belongs to $COMM(c)$, where $c = [\log_2 s] + 1$.

For each natural n, we consider the protocol $D_n = (\Pi_n, \Phi_n)$, where Π_n divides the set $\{1, 2, \ldots, 2n\}$ into the sets $S_I = \{1, \ldots, n\}$ and $S_{II} = \{n + 1, \ldots, 2n\}$, and for all x in $\{0, 1\}^n$ and all j in $\{1, \ldots, s\}$.

$\Phi_n(x) = BIN_c(i)$ iff A computing on x ends in the state p_i,

$\Phi_n(x, BIN_c(j)) = accept\ (reject)$ iff A beginning to compute on x in the state p_j ends the computation in an accepting (unaccepting) state.

It is easy to see that D_n accepts the input word iff A accepts this word, which proves our assertion.

Considering the assertion of Theorem 1 the natural question is, whether there exists a constant m such that $\mathscr{R} \subseteq COOM(m)$. In the following we show that such a constant does not exist.

Theorem 2. For all natural c there exists L in $\mathscr{R}$ such that $L \notin COMM(c)$.

Proof. We shall consider the language $L = \{x \in \{0, 1\}^*, \# 0(x) = 2^{c+1}\}$. We prove by contradiction that for $n = 2^{c+1}$ c communication bits do not suffice for recognizing $L_n = L \cap \{0, 1\}^{2n}$.

Let there exist a protocol $D_n = (\Pi, \Phi)$ recognizing L_n within communication c. Let us divide all words in $\{0, 1\}^n$ into $n + 1 = 2^{c+1} + 1$ classes $K_0, K_1, \ldots, K_n$, where $K_i = \{x \in \{0, 1\}^n | \# 0(x) = i\}$. Clearly, for each y_I (y_{II}) in K_i $(0 \leqslant i \leqslant m)$ and each u_{II} (u_I) in K_{m-i} the input word $\Pi^{-1}(y_I, u_{II})$ ($\Pi^{-1}(y_{II}, u_I)$) belongs to L.

Since the number of all accepting computations of D_n is at most 2^c, there exist two input words $\Pi^{-1}(y_k, u_k)$ and $\Pi^{-1}(y_m, u_m)$ in L having the same accepting computation, where $y_k \in K_k$, $y_m \in K_m$, $u_k \in K_{n-k}$, $u_m \in K_{n-m}$ and $k \neq m$. So we have a contradiction because D_n accepts the words $\Pi^{-1}(y_k, u_m)$ and $\Pi^{-1}(y_m, u_k)$ which do not belong to L.

Considering Theorems 1 and 2 we obtain $\mathscr{R} \subseteq \bigcup_{c=1}^{\infty} COMM(c)$. Let us consider the question whether the equality holds in this relation. In what follows we shall show that $\mathscr{R} \subset \bigcup_{c=1}^{\infty} COMM(c)$, especially we shall prove the more powerful result that there exists such a language L_I in $COMM(1)$ that it cannot be generated by any context grammar.

Theorem 3. There exists a language L_I in $COOM(1)$ such that L_I is not in $\mathscr{L}_{CS}$ that is the family of all context sensitive languages.

Proof. Let $x_1, x_2, x_3, \ldots$ be the infinite sequence of all words in $\{0, 1\}^*$, which is lexicographically arranged. Let $T_1, T_2, T_3, \ldots$ be the infinite sequence of all context grammars arranged according to their binary coding. It is well known that the language $L = \{x_i | x_i$ cannot be derived in $T_i\}$ does not belong to $\mathscr{L}_{CS}$. We shall consider the language $L_I = \{a_1 1 a_2 1 a_3 1 \ldots a_k 1 | k \geqslant 1, a_i \in \{0, 1\}$ for all $i = 1, \ldots, k$, and $a_1 a_2 a_3 \ldots a_k \in L\}$. Let there be a context grammar T' generating L_I. Then it is no problem to construct context grammar T accepting L, which implies L_I is not in $\mathscr{L}_{CS}$. Now, we shall show that L_I is in $COMM(1)$.

We construct, for each natural n, the protocol $D_n = (\Pi_n, \Phi_n)$ recognizing $L \cap \{0, 1\}^{2n}$, where Π_n is the partition of $\{1, 2, \ldots, 2n\}$ into the sets $S_I = \{2k | 1 \leqslant k \leqslant n\}$ and $S_{II} = \{2k + 1 | 0 \leqslant k \leqslant n - 1\}$, and Φ_n is defined in the following way. For all x in $\{0, 1\}^n$:

$$
\begin{aligned}
&\Phi_n(x) = 1 && \text{iff } x = 1^n \\
&\Phi_n(x) = reject && \text{iff } x \neq 1^n \\
&\Phi_n(x, 1) = accept && \text{iff } x \in L_I \\
&\Phi_n(x, 1) = reject && \text{iff } x \notin L_I.
\end{aligned}
$$

Obviously, D_n accepts the language $L \cap \{0, 1\}^{2n}$ for all n.

Considering the results obtained we could make the following reflection. Either most languages of the Chomsky hierarchy families can be recognized in constant communication complexity (i.e. the languages of Chomsky hierarchy families are involved in the simplest communication complexity families), or there exists a simple language according to Chomsky hierarchy which is not simple according to communication complexity hierarchy (i.e. the hierarchies considered are uncomparable). We shall show that the second part of the consideration introduced holds.

Theorem 4. There exists deterministic context-free language L' which does not belong to $COMM([\log_2 n] - 1)$.

Proof. Let us consider the language $L' = \{x \in \{0, 1\}^* | \# 1(x) = \# 0(x)\}$. It can be easily seen that it is no problem to construct a one-way deterministic counter automaton recognizing L'. So, L' is deterministic context-free language.

Clearly, we can write $L' = \bigcup_{n=1}^{\infty} L_n$, where $L_n = \{x \in \{0, 1\}^* | |x| = 2n$ and $\# 0(x) = n\}$. In the proof of Theorem 2 we showed that the language

$$\{x \in \{0, 1\}^* | \# 0(x) = 2^{c+1}\} \cap \{0, 1\}^{2n}$$

requires, for $n = 2^{c+1}$, communication complexity greater than c. So we have that $L_n = L' \cap \{0, 1\}^{2n}$ cannot be recognized in communication complexity $[\log_2 n] - 1$ for all n, therefore $L' \notin COMM([\log_2 n] - 1)$.

We conclude this paper with the note that no substantial coherence is between the Chomsky hierarchy and the communication complexity hierarchy. Howewer, this does not exclude the possiblity of some relation between the Chomsky hierarchy and the layout area of the chips recognizing the languages.

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РЕЗЮМЕ

СООТНОШЕНИЕ МЕЖДУ ИЕРАРХИЕЙ ЧОМСКОГО И ИЕРАРХИЕЙ КОММУНИКАТИВНОЙ СЛОЖНОСТИ

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В работе иследована новая мера сложности определенна следовательно. Пусть язык $L \subseteq (\{0, 1\}^2)^*$ распознается двумя отдаленными вычислительными машинами. Каждая машина получает половину вводных битов и вычисление осуществляется при помощи протоколов передачи данных между этими машинами. Минимальное количество битов, которыми машины должны обменятся, чтобы распознать $L \cap \{0, 1\}^{2n}$, разделенный во всех частях вводных битов на две одинаковые доли, называется коммуникативной сложностью языка L. В работе сровнена иерархия коммуникативной сложности с иерархией Чомского.

SÚHRN

VZŤAH CHOMSKÉHO HIERARCHIE A HIERARCHIE KOMUNIKAČNEJ ZLOŽITOSTI

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V práci sa študuje nová miera zložitosti pre rozpoznávanie jazykov definovaná nasledovným spôsobom. Predpokladajme, že jazyk $L \subseteq (\{0, 1\}^2)^*$ má byť rozpoznávaný dvoma rôznymi počítačmi. Každý počítač dostane polovičku vstupných bitov a výpočet prebieha používajúc protokoly pre komunikáciu medzi týmito dvoma počítačmi. Minimálny počet bitov, ktoré musia byť vymenené za účelom rozpoznania $L \cap \{0, 1\}^{2n}$, minimalizovaný cez všetky rozdelenia vstupu na dve rovnako veľké časti, a uvažovaný ako funkcia n, sa nazýva komunikačná zložitosť jazyka L. V práci je ukázaný vzťah medzi hierarchiou komunikačnej zložitosti a Chomského hierarchiou.