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BAD EXAMPLES OF THE METRIC TRAVELING SALESMAN PROBLEM FOR THE 2-CHANGE HEURISTIC

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Given a complete (undirected) graph K_n (on n vertices) where each edge ij has a length $c_{ij} \geqslant 0$, the traveling salesman problem (TSP) asks to find a shortest tour Z of K_n (i.e. Z is such a hamiltonian cycle of K_n for which the sum of the lengths of its edges is the least possible). If the lengths c_{ij} fulfill the triangle inequality the problem is referred to as the metric TSP. There exist many exact or approximate procedures (heuristics) for the solution of TSP's (see e.g. [3] for a survey). As a measure of the approximation for a given heuristic H and an example, one can consider the ratio $c(Z_H)/c(Z^*)$ where $c(Z^*)$ and $c(Z_H)$ are the lengths of an optimal tour Z^* and that tour Z_H provided by H, respectively. In [14] several $O(n^2)$ heuristic are analyzed (they are running in time n^2) ensuring the ratio not exceeding 2 for any instance of the metric TSP. The best known ratio is 3/2 and it is provided by the Christofides $O(n^3)$ heuristic [2]. But no smaller ratio is guaranteed even in the case when the metric is restricted to rectilinear or Euclidean distances in the plane [4]. However, no polynomial time algorithm is expected for ratio one, i.e. for finding an exact solution, because it is known that the metric (as well as rectilinear and Euclidean in the plane) TSP is NP-hard (see e.g. [7]). The general (non-metric) TSP is known to be much harder. More precisely, for any number r, the problem of finding an approximate solution with ratio not exceeding r is NP-hard [15]. There are many other heuristics for TSP (see e.g. [14, 6, 8, 9]); the last two papers contain also the empirical analysis of such heuristics.

One of heuristics uses the following idea. Let $k \geqslant 2$ be an integer. Given a tour Z, remove k edges from Z and then replace them with another k edges to obtain a new tour Z'. This operation is called the k-change. In the k-change heuristic, starting from a given tour Z^0, we successively form tours Z^1, Z^2, ..., where Z^{i+1} arise from Z^i by a k-change such that $c(Z^{i+1}) < c(Z^i)$ for every i. Since there are only finitely many tours, the sequence is finite; the last member is called a k-optimal tour. One iteration of this method (finding Z^{i+1} or verifying that Z^i is k-optimal) can be done in time $O(n^k)$.

The k-change method appeared in several papers: Croes [5] used $k = 2$ and Bock [1] used $k = 3$, but it was Lin [10] who first convincingly demonstrated the power of the 3-change approach. Later, Lin and Kerninghan [11] described an improved procedure based especially on 3-change. Another modified 3-change procedure has been proposed by Or [12]. All these contributions have reported the successful application of such a local search. Experiments show that a k--optimal tour is not necessarily optimal. Lin [10] found empirically that 3-optimal solutions are much better than 2-optimal solutions, but that 4-optimal solutions are not sufficiently better than 3-optimal solutions to justify the additional running time.

On the other hand, there are also some theoretical results on k-optimal solutions. Papadimitriou and Steiglitz [13] constructed examples of non-metric TSP with $n = 8p$ vertices that have the following property: There is exactly one optimal tour with length n, and there are $2^{p-1}(p-1)!$ tours that are next-best, have arbitrarily large length, and cannot be improved by changing fewer than $3p$ edges. Even more catastrophic examples are available in the non-symmetric case (digraphs) [13]. In the case of the metric TSP they gave no such examples, but showed that for a second-best tour Z' and a best tour Z^* one has $c(Z')/c(Z^*) \leqslant$ $\leqslant 1 + 2/n$. Rosenkrantz, Stearns and Lewis [14] have shown that for each $n \geqslant 8$ there exists an example of the metric TSP having a k-optimal tour Z for all $k \leqslant n/4$ with $c(Z)/c(Z^*) = 2(1 - 1/n)$.

The aim of this note is to show that also for the metric TSP there is a class of examples where a 2-optimal tour Z admits the ratio $c(Z)/c(Z^*)$ to be arbitrarily large. This result partially explains why sometimes 2-optimal solutions provide only weak approximations.

Theorem. For any number $r > 1$ there exists an example of the metric traveling salesman problem possessing a 2-optimal tour whose length is at least r times the length of an optimal tour.

Proof. Let us consider the following graph $G_{s,t}$ (see *Fig. 1*) where s and t are positive integer parameters which will be specified later on. We define $G_{s,t}$ by describing the vertex set and the edge set:

$$V(G_{s,t}) = \{u_{0t}\} \cup \bigcup_{i=1}^{s} \bigcup_{j=1}^{t} \{u_{ij}, v_{ij}\};$$

$E(G_{s,t})$ consists of the edges of two edge-disjoint hamiltonian cycles Z_1 and Z_2 of $G_{s,t}$, where

$$E(Z_1) = \bigcup_{i=1}^{s} \left[\{u_{i-1,t}v_{it}\} \cup \bigcup_{j=1}^{t} \{v_{i,t+1-j}u_{ij}\} \cup \bigcup_{j-1}^{t-1} \{u_{ij}v_{i,t-j}\} \right] \cup \{u_{st}u_{0t}\},$$

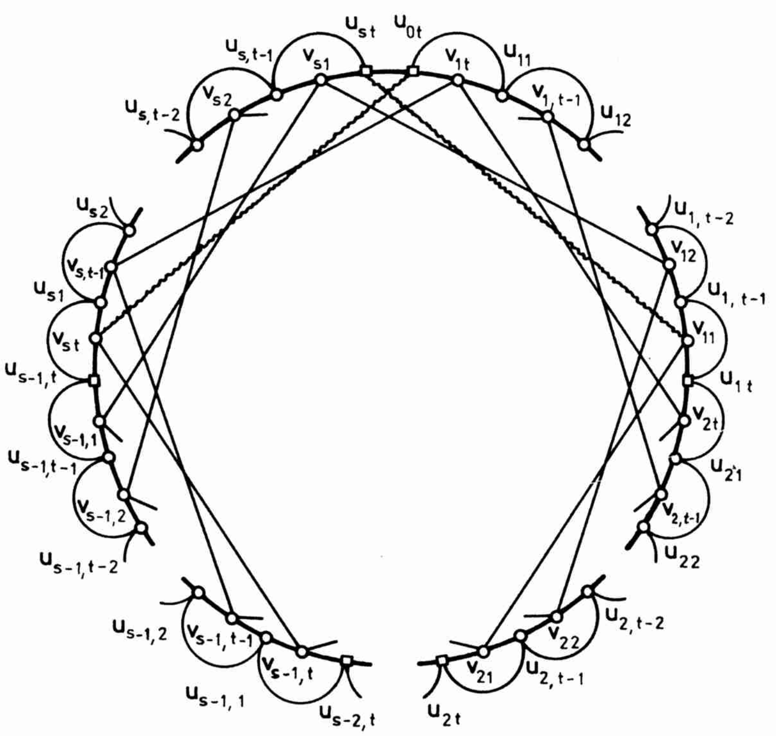

Fig. 1

$$E(Z_2) = \{u_{0t}u_{11}\} \cup \bigcup_{i=1}^{s}\bigcup_{j=1}^{t-1}\{u_{ij}u_{i,j+1}\} \cup \{u_{st}v_{11}\} \cup$$

$$\cup \bigcup_{i=1}^{s-1}\bigcup_{j=1}^{t-1}\{v_{ij}v_{i+1,j},\ v_{sj}v_{1,j+1},\ v_{it}v_{i+1,t}\} \cup \{v_{st}u_{0t}\}.$$

Hence $G_{s,t}$ has $n = 2st + 1$ vertices.

Let b be an integer with $b \geqslant r + 1$ and assume that $s \geqslant t \geqslant 2b - 1$. We define the lengths of the edges of $G_{s,t}$ as follows. Every edge of Z_1 (see fat lines in *Fig. 1*) as well as the edges $u_{st}v_{11}$ and $v_{st}u_{0t}$ (wave lines) have length b; each of the remaining edges of Z_2 has length one. Thus the lengths of Z_1 and Z_2 are nb and $n - 2 + 2b$, respectively.

Now, let us consider the complete graph K_n on the vertex set $V(G_{s,t})$, where every edge xy of K_n has length equal to the distance $d(x, y)$ of x and y in $G_{s,t}$. One can easily verify that every edge of $G_{s,t}$ has the same length in K_n as in $G_{s,t}$, because s and t are sufficiently large relative to b. Thus we have ensured the triangle inequality.

We are going to prove that the cycle Z_1 is a 2-optimal tour in K_n, or in other words, that no 2-change produces a shorter tour than Z_1. Any 2-change has the form depicted in *Fig. 2*, where the edges of the cycle Z_1 are depicted by solid lines. We delete edges $e_1 = x_1y_1$ and $e_2 = x_2y_2$ (crossed lines) and add $f_1 = x_1y_2$ and $f_2 = y_1x_2$ (dashed lines). Owing to the reasons of symmetry, we consider only the following three cases.

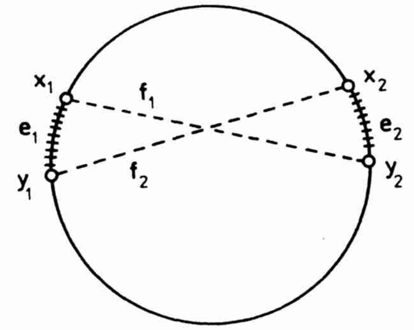

Fig. 2

Case 1: $x_1 = u_{s,t}$, $y_1 = v_{s1}$, $x_2 = u_{1,t-1}$, $y_2 = v_{11}$. Then f_1 has length b and f_2 has length $1 + b$. Therefore the new cycle is longer.

Case 2: $x_1 = v_{1t}$, $y_1 = u_{0t}$, $x_2 = u_{1t}$, $y_2 = v_{2t}$. The length of f_1 is t and that of f_2 is one. Since $t + 1 \geqslant 2b$, the new cycle is not shorter.

Case 3: $x_1 = u_{0t}$, $y_1 = u_{st}$, $x_2 = v_{11}$, $y_2 = u_{1t}$. Then f_1 has length b and f_2 has length t. Hence the new cycle is longer.

The other cases are similar and can be left to the reader. Thus the cycle Z_1 is a 2-optimal tour. The ratio of the lengths of Z_1 and Z_2 is $nb/(n - 2 + 2b)$ and it runs to b if n runs to infinity (e.g. let $s = t \to \infty$). As the cycle Z_2 is at least as long as an optimal tour and $b \geqslant r + 1$, Theorem is proved.

Note that we were unsuccessful in proving or disproving an analogical assertion for k-optimality with $k \geqslant 3$. Our experiences led to the following conjecture.

Conjecture. For every integer $k \geqslant 3$ and any example of the metric TSP the ratio of the lengths of a k-optimal tour and an optimal tour does not exceed 2. (Recall that there are examples with the ratio equal to $2 - 2/n$ [14].)

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SÚHRN

ZLÉ PRÍKLADY METRICKEJ ÚLOHY OBCHODNÉHO CESTUJÚCEHO PRE 2-VÝMENNÚ HEURISTIKU

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Pre ľubovoľné číslo *r* je daný príklad metrickej úlohy obchodného cestujúceho, v ktorom existuje taká 2-optimálna pochôdzka, že podiel jej dĺžky a dĺžky optimálnej pochôdzky presahuje *r*.

РЕЗЮМЕ

ПЛОХИЕ ПРИМЕРЫ МЕТРИЧЕСКОЙ ПРОБЛЕМЫ КОММИВОЯЖЕРА ДЛЯ 2-ЗАМЕННОЙ ЭВРИСТИКИ

Ян Плесник, Братислава

Для любого числа *r* дается пример метрической задачи коммивояжера, у которого существует такой 2-оптимальный цикл, что отношение его длины и длины оптимального цикла превосходит *r*.

KYBERNETIKA
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