UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XLVIII—XLIX — 1986

INTEGRAL EQUIVALENCE OF AN ORDINARY AND A FUNCTIONAL DIFFERENTIAL EQUATION

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In [2] a new notion, the (ψ, p)-integral equivalence of two systems of ordinary differential equations is introduced and some sufficient conditions for (ψ, p)-integral equivalence are found. In this paper we will prove an (ψ, p)-integral equivalence theorem for the systems

$$u' = F(t, u_t) \tag{a}$$

and

$$v' = G(t, v). \tag{b}$$

Suppose that F and G are such that they guarantee the existence of solutions of (a) and (b), respectively, on the infinite interval $\langle 0, \infty)$.

Definition 1. Let $\psi(t)$ be a positive continuous function on an interval $\langle t_0, \infty)$ and let $p > 0$. We shall say that two systems (a) and (b) are (ψ, p)-integral equivalent on $\langle t_0, \infty)$ iff for each solution $u(t)$ of (a) there exists a solution $v(t)$ of (b) such that

$$\Psi^{-1}(t)|u(t) - v(t)| \in L_p(t_0, \infty) \tag{c}$$

and conversely, for each solution $v(t)$ of (b) there exists a solution $u(t)$ of (a) such that (c) holds. By a restricted (ψ, p)-integral equivalence between (a) and (b) we shall mean that the relation (c) is satisfied for some subsets of solutions of (a) and (b), e.g. for the bounded solutions.

We will say that a function $z(t)$ is ψ-bounded on the interval $\langle t_0, \infty)$ iff

$$\sup_{t \geq t_0} |\psi^{-1}(t) z(t)| < \infty.$$

Next we will consider the systems

$$u'(t) = A(t)u(t) + F(t, u_t) \tag{1}$$

and

$$v'(t) = A(t)v(t), \tag{2}$$

where $A(t)$ is an $n \times n$ matrix-function defined on $\langle 0, \infty)$, whose elements are integrable on compact subsets of $\langle 0, \infty)$; u and v are n-dimensional vectors and $F(t, \Phi)$ is a function on $\langle 0, \infty) \times C_\tau$ into R^n, where C_τ is a space of continuous functions on $\langle -\tau, 0\rangle$ (for some $\tau > 0$) into R^n with the norm

$$|\Phi| = \sup_{-\tau \leqslant \Theta \leqslant 0} |\Phi(\Theta)|.$$

$|.|$ denotes any convenient vector (matrix) norm.

If $u(t)$ is any function on $\langle t_0 - \tau, \infty)$ into R^n, then for each t in $t_0 \leqslant t < \infty$ the symbol u_t denotes the element of C_τ defined by

$$u_t(\Theta) = u(t + \Theta), \qquad -\tau \leqslant \Theta \leqslant 0.$$

We shall need the following lemmas in our considerations:

Lemma 1. (Lemma 1, A. Haščák [1].) Let $p \geqslant 1$ and $g(t) \geqslant 0$ be continuous on $0 \leqslant t < \infty$ such that

$$\int_0^\infty s^{1/p} g(s)\, \mathrm{d}s < \infty.$$

Then

$$\int_t^\infty g(s)\, \mathrm{d}s \in L_{p'}(0, \infty), \qquad p' \geqslant p.$$

Lemma 2. (Lemma 3, A. Haščák, M. Švec [2].) Let $\psi(t)$ and $\varphi(t)$ be positive functions for $t \geqslant 0$, $V(t)$ a nonsingular matrix and P a projection. Further, suppose that

$$\left[\int_0^t |\psi^{-1}(t) V(t) P V^{-1}(s) \varphi(s)|^p\, \mathrm{d}s\right]^{1/p} \leqslant K$$

for $t \geqslant 0$, $K > 0$, $p > 0$ and

$$\int_0^\infty \exp\left(-K^{-p} \int_0^t \varphi^p(s) \psi^{-p}(s)\, \mathrm{d}s\right) \mathrm{d}t < \infty.$$

Then

$$\lim \psi^{-1}(t) |V(t)P| = 0 \quad \text{as} \quad t \to \infty$$

and

$$|\psi^{-1}(t) V(t) P| \in L_p(0, \infty).$$

Now we are going to prove the following theorem:

Theorem 1. Let $V(t)$ be a fundamental matrix of (2), $\psi(t)$ and $\varphi(t)$ positive continuous functions for $t > -\infty$ and $t \geqslant 0$ respectively. Suppose that:

a) there exist supplementary projectors P_1, P_2, a constant $K > 0$ and $2 \leqslant p > \infty$ such that

$$\int_0^t |\psi^{-1}(t)V(t)P_1V^{-1}(s)\varphi(s)|^p \,\mathrm{d}s + \int_t^\infty |\varphi^{-1}(t)V(t)P_2V^{-1}(s)\varphi(s)|^p \,\mathrm{d}s \leqslant K^p$$

for all $t \geqslant 0$,

b) for each $M > 0$ there is $\tau > 0$ such that $F(t, \Phi)$ is defined for Φ in C_τ, $|\psi^{-1}\Phi| \leqslant M$, $t \geqslant 0$. Further, let $F(t, u_t)$ be a continuous function of t for $t \geqslant 0$ if $u(t)$ is a continuous function on $-\tau \leqslant t < \infty$ with $|\psi^{-1}(t)u(t)| \leqslant M$. If $u^{(n)} \to u$ in the sense of uniform convergence on each of the compact subsets of $\langle 0, \infty)$, then $F(s, u_s^{(n)}) \to F(s, u_s)$ uniformly on each compact subset of $\langle 0, \infty)$;

c) there exists g: $\langle 0, \infty) \times \langle 0, \infty) \to \langle 0, \infty)$ such that (i) $g(t, u)$ is monotone nondecreasing in u for each fixed $t \in \langle 0, \infty)$ and integrable on compact subsets of $\langle 0, \infty)$ for fixed $u \in \langle 0, \infty)$, (ii) $\int_0^\infty s^{1/p} g^{p'}(s, c)\,\mathrm{d}s < \infty$ for any constant $c \geqslant 0$, where $1/p + 1/p' = 1$, (iii) for any $M > 0$ and corresponding τ

$$|F(t, \Phi)| \leqslant \varphi(t) g(t, \psi^{-1}(t)|\Phi|)$$

for every Φ in C_τ with $|\psi^{-1}\Phi| \leqslant M$ a.e. on $\langle 0, \infty)$,

d) $$\int_0^\infty \exp\left\{-K^{-p}\int_0^t \varphi^p(s)\psi^{-p}(s)\,\mathrm{d}s\right\}\mathrm{d}t < \infty$$

e) $$\int_0^\infty |P_1V^{-1}(s)\varphi(s)| g(s, c)\,\mathrm{d}s < \infty.$$

Then the sets of ψ-bounded solutions of (1) and of (2) are (ψ, p)-integral equivalent.

Proof. Let $v(t)$ be a ψ-bounded solution of (2) on $\langle t_0, \infty)$, $t_0 \geqslant 0$. Choose $\varrho > 0$ and M so that $|\psi^{-1}(t)v(t)| \leqslant \varrho$ and $M \geqslant 3\varrho$. By hypotheses b) there is a positive number τ such that $F(t, \Phi)$ is defined for $\Phi \in C_\tau$, $|\psi^{-1}\Phi| \leqslant M$, $t \geqslant 0$. For the rest the proof ϱ, M and τ are fixed. Choose t_0 so large that

$$K\left\{\int_{t_0}^\infty g^{p'}(s, 3\varrho)\,\mathrm{d}s\right\}^{1/p'} \leqslant \varrho. \tag{3}$$

Let

$$B_{\psi,\varrho} = \{u\colon u \text{ is continuous on } \langle t_0 - \tau, \infty) \text{ and } \sup_{t \geqslant t_0 - \tau} |\psi^{-1}(t)u(t)| \leqslant \varrho\}$$

Definer for $u \in B_{\psi,3\varrho}$ the operator

$$Tu(t) = \begin{cases} v(t_0) & \text{for } t_0 - \tau \leqslant t \leqslant t_0 \\ v(t) + \int_{t_0}^{t} V(t)P_1V^{-1}(s)F(s, u_s)\,\mathrm{d}s - \\ \quad - \int_t^{\infty} V(t)P_2V^{-1}(s)F(s, u_s)\,\mathrm{d}s, & t \geqslant t_0 \end{cases}$$

Since $u(t)$ is continuous on $t \geqslant t_0 - \tau$ and

$$|\psi^{-1}(t)u(t)| \leqslant 3\varrho \leqslant M,$$

it follows from b) that $F(t, u_t)$ is continuous for $t \geqslant t_0$. Also $u \in B_{\psi,3\varrho}$ implies that $u_t \in C_\tau$ for $t \geqslant t_0$ with

$$\psi^{-1}(t)|u_t| \leqslant 3\varrho \leqslant M.$$

Now the existence of $\int_t^{\infty} V(t)P_2V^{-1}(s)F(s, u_s)\,\mathrm{d}s$ is guaranteed by a) and c). Then

$$|\psi^{-1}(t)Tu(t)| \leqslant \psi^{-1}(t)|v(t)| + \int_{t_0}^{t} |\psi^{-1}(t)V(t)P_1V^{-1}(s)F(s, u_s)|\,\mathrm{d}s +$$

$$+ \int_t^{\infty} |\psi^{-1}(t)V(t)P_2V^{-1}(s)F(s, u_s)|\,\mathrm{d}s \leqslant$$

$$\leqslant \varrho + \int_{t_0}^{t} |\psi^{-1}(t)V(t)P_1V^{-1}(s)\varphi(s)|g(s, \psi^{-1}(s)|u_s|)\,\mathrm{d}s +$$

$$+ \int_t^{\infty} |\psi^{-1}(t)V(t)P_2V^{-1}(s)\varphi(s)|g(s, \psi^{-1}(s)|u_s|)\,\mathrm{d}s.$$

Using the Hölder inequality, a), c) and (3) we get

$$|\psi^{-1}(t)Tu(t)| \leqslant$$

$$\leqslant \varrho + \left\{\int_{t_0}^{t} |\psi^{-1}(t)V(t)P_1V^{-1}(s)\varphi(s)|^p\,\mathrm{d}s\right\}^{1/p} \left\{\int_{t_0}^{t} g^{p'}(s, 3\varrho)\,\mathrm{d}s\right\}^{1/p'} +$$

$$+ \left\{\int_t^{\infty} |\psi^{-1}(t)V(t)P_2V^{-1}(s)\varphi(s)|^p\,\mathrm{d}s\right\}^{1/p} \left\{\int_t^{\infty} g^{p'}(s, 3\varrho)\,\mathrm{d}s\right\}^{1/p'} \leqslant$$

$$\leqslant \varrho + K\left\{\int_{t_0}^{\infty} g^{p'}(s, 3\varrho)\,\mathrm{d}s\right\}^{1/p'} \leqslant 2\varrho.$$

Thus T maps $B_{\psi,3\varrho}$ into itself.

Next we are going to prove the continuity of T on $B_{\psi, 3\varrho}$. Let $u^{(n)}(t)$, $u(t) \in B_{\psi, 3\varrho}$, $u^{(n)}(t)$ converges to $u(t)$ uniformly on compact subintervals of $\langle t_0, \infty)$. We have

$$|Tu^{(n)}(t) - Tu(t)| \leqslant$$

$$\leqslant \int_{t_0}^{t} |V(t)P_1V^{-1}(s)|\,|F(s, u_s^{(n)}) - F(s, u_s)|\,\mathrm{d}s +$$

$$+ \int_{t}^{\infty} |V(t)P_2V^{-1}(s)|\,|F(s, u_s^{(n)}) - F(s, u_s)|\,\mathrm{d}s =$$

$$= \psi(t)\int_{t_0}^{t} |\psi^{-1}(t)V(t)P_1V^{-1}(s)\varphi(s)|\,\varphi^{-1}(s)\,|F(s, u_s^{(n)}) - F(s, u_s)|\,\mathrm{d}s +$$

$$+ \psi(t)\int_{t}^{\infty} |\psi^{-1}(t)V(t)P_2V^{-1}(s)\varphi(s)|\,\varphi^{-1}(s)\,|F(s, u_s^{(n)}) - F(s, u_s)|\,\mathrm{d}s \leqslant$$

$$\leqslant \psi(t)\left\{\int_{t_0}^{t} |\psi^{-1}(t)V(t)P_1V^{-1}(s)\varphi(s)|^p\,\mathrm{d}s\right\}^{1/p} \cdot$$

$$\cdot \left\{\int_{t_0}^{t} \varphi^{-p'}(s)\,|F(s, u_s^{(n)}) - F(s, u_s)|^{p'}\,\mathrm{d}s\right\}^{1/p'} +$$

$$+ \psi(t)\left\{\int_{t}^{\infty} |\psi^{-1}(t)V(t)P_2V^{-1}(s)\varphi(s)|^p\,\mathrm{d}s\right\}^{1/p} \cdot$$

$$\cdot \left\{\int_{t}^{\infty} \varphi^{-p'}(s)\,|F(s, u_s^{(n)}) - F(s, u_s)|^{p'}\,\mathrm{d}s\right\}^{1/p'} \leqslant$$

$$\leqslant \psi(t)K\left\{\int_{t_0}^{\infty} \varphi^{-p'}(s)\,|F(s, u_s^{(n)}) - F(s, u_s)|^{p'}\,\mathrm{d}s\right\}^{1/p'} =$$

$$= K\psi(t)\left\{\int_{t_0}^{t_1} \varphi^{-p'}(s)\,|F(s, u_s^{(n)}) - F(s, u_s)|^{p'}\,\mathrm{d}s + \right.$$

$$\left. + \int_{t_1}^{\infty} \varphi^{-p'}(s)\,|F(s, u_s^{(n)}) - F(s, u_s)|^{p'}\,\mathrm{d}s\right\}^{1/p'}.$$

On $\langle t_0, t_1\rangle$ $u^{(n)}(t)$ converges to $u(t)$ uniformly. Then by b) it follows that to $\varepsilon > 0$ there is $n_0(t_1)$ such that for $n \geqslant n_0(t_1)$ we have

$$\varphi^{-1}(s)\,|F(s, u_s^{(n)}) - F(s, u_s)| < \frac{\varepsilon}{4K(t_1 - t_0)^{1/p'}} \quad \text{for } s \in \langle t_0, t_1\rangle$$

Applying this and c) (iii) we have for $n \geqslant n_0$

$$|Tu^{(n)}(t) - Tu(t)| \leqslant K\psi(t)\left[\frac{p'}{4^{p'}K^{p'}} + 2\int_{t_1}^{\infty} \varphi^{-p'}(s)\varphi^{p'}(s)g^{p'}(s, 3\varrho)\,\mathrm{d}s\right]^{1/p'}.$$

Choose t_1 such that

$$2\int_{t_1}^{\infty} g^{p'}(s, 3\varrho)\, \mathrm{d}s < \frac{\varepsilon^{p'}}{4^{p'}K^{p'}}.$$

Then we have

$$|Tu^{(n)}(t) - Tu(t)| < \varepsilon\psi(t) \quad \text{on } \langle t_0, \infty).$$

For $t \in \langle t_0 - \tau, t_0\rangle$,

$$Tu^{(n)}(t) - Tu(t) = Tu^{(n)}(t_0) - Tu(t_0).$$

This shows that T is continuous on $B_{\psi, 3\varrho}$.

The functions in $TB_{\psi, 3\varrho}$ are evidently uniformly bounded for each $t \geq t_0 - \tau$ because $TB_{\psi, 3\varrho} \subset B_{\psi, 3\varrho}$.

Because $w = Tu$ is a solution of the equation

$$w'(t) = A(t)w(t) + F(t, u_t) \quad \text{for } t \geq t_0,$$

the derivatives of the functions in $TB_{\psi, 3\varrho}$ are uniformly bounded on every compact interval. Thus the functions in $TB_{\psi, 3\varrho}$ are equicontinuous on every compact subinterval of $\langle t_0 - \tau, \infty)$.

Then from Schauder's fixed point theorem follows the existence of a fixed point $u(t)$ of T in $B_{\psi, 3\varrho}$. We have

$$u(t) = v(t) + \int_{t_0}^{t} V(t)P_1V^{-1}(s)F(s, u_s)\, \mathrm{d}s - \int_{t}^{\infty} V(t)P_2V^{-1}(s)F(s, u_s)\, \mathrm{d}s, \qquad t \geq t_0$$

$$u(t) = u(t_0), \qquad t_0 - \tau \leq t \leq t_0.$$

Also

$$u'(t) = A(t)u(t) + F(t, u_t), \qquad t \geq t_0.$$

A direct verification shows that this fixed point $u(t)$ is a ψ-bounded solution of (1).

Conversely, let $u(t)$ be a ψ-bounded solution of (1) on some interval $\langle t_0, \infty)$ which satisfies

$$|\psi^{-1}(t)u(t)| \leq M, \quad \psi^{-1}(t)\,|u_t| \leq M, \qquad t \geq t_0.$$

Then $F(t, u_t)$ is defined for $t \geq t_0$ and

$$|F(t, u_t)| \leq \varphi(t)g(t, \psi^{-1}(t)\,|u_t|) \leq \varphi(t)g(t, M).$$

Define

$$v(t) = u(t) - \int_{t_0}^{t} V(t)P_1V^{-1}(s)F(s, u_s)\, \mathrm{d}s + \int_{t}^{\infty} V(t)P_2V^{-1}(s)F(s, u_s)\, \mathrm{d}s, \qquad t \geq t_0.$$

The infinite integral converges and is bounded by

$$K\psi(t)\left[\int_t^\infty g^{p'}(s, M)\,\mathrm{d}s\right]^{1/p'}.$$

It is easy to prove that $v(t)$ is a ψ-bounded solution of (2).

Now we have to prove that $\psi^{-1}(t)\,|u(t) - v(t)| \in L_p(t_0, \infty)$. We have

$$\psi^{-1}(t)\,[u(t) - v(t)] = \int_{t_0}^t \psi^{-1}(t)V(t)P_1V^{-1}(s)F(s, u_s)\,\mathrm{d}s - $$
$$- \int_t^\infty \psi^{-1}(t)V(t)P_2V^{-1}(s)F(s, u_s)\,\mathrm{d}s.$$

It suffices to prove that the terms on the right-hand side belong to $L_p(t_0, \infty)$. Respecting the assumptions of the theorem by Hölder inequality we get

$$\left|\int_{t_0}^t \psi^{-1}(t)V(t)P_1V^{-1}(s)F(s, u_s)\,\mathrm{d}s\right| \leqslant$$
$$\leqslant |\psi^{-1}(t)V(t)P_1| \int_{t_0}^t |P_1V^{-1}(s)\varphi(s)g(s, 3\varrho)|\,\mathrm{d}s.$$

From Lemma 2 we have that

$$|\psi^{-1}(t)V(t)P_1| \in L_p(t_0, \infty)$$

and e) holds; it is evident that this first term belongs to $L_p(t_0, \infty)$. For the second term we have

$$\left|\int_t^\infty \psi^{-1}(t)V(t)P_2V^{-1}(s)F(s, u_s)\,\mathrm{d}s\right| \leqslant$$
$$\leqslant \int_t^\infty |\psi^{-1}(t)V(t)P_2V^{-1}(s)\varphi(s)|\,g(s, 3\varrho)\,\mathrm{d}s \leqslant$$
$$\leqslant \left(\int_t^\infty |\psi^{-1}(t)V(t)P_2V^{-1}(s)\varphi(s)|^p\,\mathrm{d}s\right)^{1/p} \left(\int_t^\infty g^{p'}(s, 3\varrho)\,\mathrm{d}s\right)^{1/p'} \leqslant$$
$$\leqslant K\left(\int_t^\infty g^{p'}(s, 3\varrho)\,\mathrm{d}s\right)^{1/p'}.$$

Thus from c) (iii) and Lemma 2 we get that also this term belongs to $L_p(t_0, \infty)$. The proof of the theorem is complete.

Remark. If we substitute in Theorem 1 the condition c) (ii) by the condition

$$\left(\int_t^\infty g^{p'}(s, c)\,\mathrm{d}s\right)^{1/p'} \in L_p(0, \infty)$$

and for p we assume that $1 < p < \infty$, then the conclusions of Theorem 1 hold.

Corollary 1.1. Let $p = 1$ (and thus $p' = \infty$). Assume that the assumptions of Theorem 1 are satisfied except c) (ii) which is substituted by the conditions $\lim_{t \to \infty} \gamma_c(t) = 0$ for each $c \geqslant 0$ and $\gamma_c(t) \in L_1(0, \infty)$, where $\gamma_c(t) = \sup_{s \geqslant t} g(s, c)$. Then the conclusion of Theorem 1 holds true.

Corollary 1.2. Let $p = \infty$ (and $p' = 1$). Let the condition a) of the Theorem 1 be replaced by

$$\sup_{t_0 \leqslant s \leqslant t} |\Psi^{-1}(t) V(t) P_1 V^{-1}(s) \varphi(s)| + \sup_{t < s < \infty} |\psi^{-1}(t) V(t) P_2 V^{-1}(s) \varphi(s)| \leqslant K$$

and

$$|\psi^{-1}(t) V(t) P_1 \langle \in L_v(0, \infty), \qquad v > 1$$

and let all the other assumptions of Theorem 1 hold. Then the sets of ψ-bounded solution of (1) and of (2) are (ψ, v)-integral equivalent.

It is possible to change the assumptions in Theorem 1 in such a way that we will assume more about the expression on the left side of the inequality in a) and less about the function $g(t, u)$. It holds

Theorem 2. Assume that the following hypotheses from the Theorem 1 are satisfied: a), b), c) (i), (iii). Instead of c) (ii) let

$$\int_0^\infty g^{p'}(t, c) \, \mathrm{d}t < \infty, \qquad 0 < c < \infty$$

be satisfied only; instead of d) let

$$\int_0^\infty \varphi^p(t) \psi^{-p}(t) \, \mathrm{d}t = \infty$$

be satisfied.

Finally, let the left side of the inequality a) belong to $L_1(0, \infty)$. Then the conclusion of Theorem 1 is still valid.

Proof. The proof of Theorem 2 can be made in the same manner as that of Theorem 1.

Theorem 3. Let $\psi(t)$, $\alpha(t)$ and $\beta(t)$ be positive continuous functions for $t \geqslant \tau_0 \geqslant 0$ with $\lim \psi^{-1}(t) = 0$ as $t \to \infty$ and $\beta(t)$ bounded on $\langle \tau_0, \infty)$. Let $V(t)$ be a fundamental matrix of (2).

Suppose further

a) for each $M > 0$ there is $\tau > 0$ such that $F(t, \Phi)$ is defined for Φ in C_τ, $|\psi^{-1} \Phi| \leqslant M$, $t \geqslant 0$; let $F(t, u_t)$ be a continuous function of t for $t \geqslant \tau_0 \geqslant 0$ if $u(t)$ is a continuous function on $\tau_0 - \tau \leqslant t < \infty$ with $|\psi^{-1}(t) u(t)| \leqslant M$. If $u^{(n)} \to u$ in the sense of uniform convergence on each of the compact subsets of $\langle \tau_0, \infty)$, then $F(s, u_s^{(n)}) \to F(s, u_s)$ uniformly on each compact subset of $\langle \tau_0, \infty)$;

b) there exist $w\colon \langle \tau_0, \infty) \times \langle 0, \infty) \to \langle 0, \infty)$ such that $|F(t, \Phi)| \leqslant w(t, |\Phi|)$ for $t \geqslant \tau_0$, $\Phi \in C_\tau$ and $|\psi^{-1}\Phi| \leqslant M$; $w(t, r)$ is nondecreasing in r for each fixed $t \geqslant \tau_0$; $w(t, c\psi(t))$ is integrable on compact subsets of $\langle \tau_0, \infty)$ for each $c \geqslant 0$;

c) $\displaystyle\int_{\tau_0}^{\infty} s\alpha(s)w(s, c\psi(s))\,\mathrm{d}s < \infty$ for each $c \geqslant 0$;

d) $\displaystyle\int_{\tau_0}^{t} \beta(t-s)\alpha(s)w(s, c\psi(s))\,\mathrm{d}s \in L_p(\tau_0, \infty)$ for each $c \geqslant 0$;

e) there are two supplementary projections P_1 and P_2 and a constant $c > 0$ such that

$$|V(t)P_1V^{-1}(s)\alpha^{-1}(s)| \leqslant c\beta(t-s) \quad \text{for } \tau_0 \leqslant s \leqslant t,$$

$$|V(t)P_2V^{-1}(s)\alpha^{-1}(s)| \leqslant c \quad \text{for } \tau_0 \leqslant t \leqslant s < \infty.$$

Then between the set of all ψ-bounded solutions of (1) and the set of all ψ-bounded solutions of (2) the $(1, p)$-integral equivalence holds, $p \geqslant 1$.

Proof. Let $v(t)$ be a ψ-bounded solution of (2) defined on $\langle t_0, \infty)$. Choose $\varrho > 0$ and M so that $|\psi^{-1}(t)v(t)| \leqslant \varrho$ and $M \geqslant 3\varrho$. By hypotheses a) there is a positive number τ such that $F(t, \Phi)$ is defined for $\Phi \in C_\tau$, $|\psi^{-1}\Phi| \leqslant M$. Let

$$\varphi(t) = \alpha^{-1}(t), \quad g(t, r) = \alpha(t)w(t, \psi(t)r) \quad \text{for } t \geqslant t_0.$$

Then for $t \geqslant t_0$, $g(t, r)$ is monotone nondecreasing in r and for each fixed $r \in \langle 0, \infty)$ $g(t, r)$ is integrable on compact subsets of $\langle t_0, \infty)$ by condition b).

From c) we get that

$$\int_{t_0}^{\infty} sg(s, c)\,\mathrm{d}s < \infty \quad \text{for } c \geqslant 0.$$

Moreover, condition e) and the assumptions on $\Psi(t)$ and $\beta(t)$ imply that there exists K such that

$$\sup_{t_0 \leqslant s \leqslant t} |\psi^{-1}(t)V(t)P_1V^{-1}(s)\varphi(s)| + \sup_{t \leqslant s < \infty} |\psi^{-1}(t)V(t)P_2V^{-1}(s)\varphi(s)| < K.$$

Now the same reasoning as in the proof of Theorem 1 with $p = \infty$ gives the existence of solution $u(t)$ of (1) such that

$$\sup_{t \geqslant t_0} |\psi^{-1}(t)u(t)| \leqslant 3\varrho$$

and

$$u(t) = v(t) + \int_{t_0}^{t} V(t)P_1V^{-1}(s)F(s, u_s)\,\mathrm{d}s - \int_{t}^{\infty} V(t)P_2V^{-1}(s)F(s, u_s)\,\mathrm{d}s, \qquad t \geqslant t_0.$$

Conversely, given a ψ-bounded solution $u(t)$ of (1) we have that

$$v(t) = u(t) - \int_{t_0}^{t} V(t)P_1V^{-1}(s)F(s, u_s)\,ds + \int_{t}^{\infty} V(t)P_2V^{-1}(s)F(s, u_s)\,ds, \qquad t \geqslant t_0$$

is a solution of (2).

Now, we have to prove that

$$|u(t) - v(t)| \in L_v(t_0, \infty).$$

We have

$$|u(t) - v(t)| \leqslant \int_{t_0}^{t} |V(t)P_1V^{-1}(s)|\,|F(s, u_s)|\,ds + \int_{t}^{\infty} |V(t)P_2V^{-1}(s)|\,|F(s, u_s)|\,ds \leqslant$$

$$\leqslant \int_{t_0}^{t} |V(t)P_1V^{-1}(s)\alpha^{-1}(s)|\,\alpha(s)w(s, 3\varrho\psi(s))\,ds + \int_{t}^{\infty} |V(t)P_2V^{-1}(s)\alpha^{-1}(s)|\,\alpha(s)w(s, 3\varrho\psi(s))\,ds \leqslant$$

$$\leqslant c\int_{t_0}^{t} \beta(t-s)\alpha(s)w(s, 3\varrho\psi(s))\,ds + c\int_{t}^{\infty} \alpha(s)w(s, 3\varrho\psi(s))\,ds.$$

It is sufficient to prove that each of the two terms on the right-hand side belongs to $L_v(t_0, \infty)$.

For the first term the statement is true by assumption d) and for the second term it follows from c) and Lemma 1.

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Received: 29. 6. 1984

SÚHRN

INTEGRÁLNA EKVIVALENCIA OBYČAJNEJ A FUNKCIONÁLNEJ DIFERENCIÁLNEJ ROVNICE

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V práci sa vyšetrujú postačujúce podmienky (ψ, p)-integrálnej ekvivalencie systémov diferenciálnych rovníc tvaru $u' = F(t, u_t)$ a $v' = G(t, v)$. Na odvodenie výsledkov sa používa variácia konštánt a veta o pevnom bode.

РЕЗЮМЕ

ИНТЕГРАЛЬНАЯ ЭКВИВАЛЕНТНОСТЬ ОБЫКНОВЕННОГО И ФУНКЦИОНАЛЬНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЙ

Александер Хащак, Братислава

В статье исследуйются достаточные условия для (ψ, p)-интегральной эквивалентности систем дифференциальных уравнений вида $u' = F(t, u_t)$, $v' = G(t, v)$. К достижению результатов используется вариация постиянных и теорема о неподвижной точке.