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ON ONE METHOD OF A POSTERIORI ESTIMATION

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1. Introductory formulations

Let Ω be a bounded domain in R^m with the bound $\partial\Omega$ which satisfies several conditions of smoothness. Let on $\bar{\Omega}$ be given a boundary value problem (see [3]) such that $u_0 \in V$ is its weak solution if

$$\forall\, v \in V\colon ((v, u_0)) = (v, f)_{L_2} + \varkappa(v, h) - ((v, w)). \tag{1}$$

$((,))$ is a bounded bilinear form, $V \subset W_2^k(\Omega)$, $(,)_{L_2}$ is a scalar product in $L_2(\Omega)$, $w \in W_2^k(\Omega)$,

$$\varkappa(v, h) = \sum_{p=1}^{r} \sum_{l=1}^{k-\mu_p} \int_{\partial\Omega_p} \frac{\partial^{t_{pl}} v}{\partial n^{t_{pl}}} h_{pl}\, \mathrm{d}S,$$

$f \in L_2(\Omega)$, $h_{pl} \in L_2(\partial\Omega_p)$ (all is real), n is *a* direction of an outer normal to Ω $\partial\Omega = \bigcup_p \partial\Omega_p$. Let

$$\forall\, v \in V\colon ((v, v)) \geqslant \alpha^2 \|v\|_V^2, \qquad \alpha \in R^1, \tag{2}$$

$$\forall\, u, v \in V\colon ((u, v)) = ((v, u)) \tag{3}$$

hold. Then there exists $u_0 \in V$ such that (1) is fulfilled and u_0 minimizes in V the functional

$$F(v) = ((v, v)) - 2(v, f)_{L_2} - 2\varkappa(v, h) + 2((v, w)),$$

while

$$F(u_0) = -((u_0, u_0)).$$

Assume that we can find the lower estimation d for the value $F(u_0)$, then

$$d \leqslant F(u_0).$$

Then from (2) we get

$$\forall v \in V\colon \alpha^2 \|u_0 - v\|_V^2 \leqslant ((u_0 - v, u_0 - v)),$$

from which, as u_0 fulfils (1),

$$\forall v \in V\colon \alpha^2 \|u_0 - v\|_V^2 \leqslant ((u_0, u_0)) - 2(u, f)_{L_2} - 2\varkappa(v, h) + 2((v, w)) + \\ + ((v, v)) = F(v) - F(u_0).$$

Thus

$$\forall v \in V\colon \|u_0 - v\|_V \leqslant \frac{1}{\alpha} (F(v) - d)^{1/2}.$$

2. The construction of lower estimation

In literature [e.g. 1] several constructions of the lower estimation d are described. But the majority of them is applicable for the single special problems only.

Theorem 1. Let

$$\forall u \in V, \ \forall v \in W_2^k(\Omega)\colon ((u - v, u - v)) \geqslant 0. \tag{4}$$

Let, further, $v_1 \in W_2^k(\Omega)$ be such that

$$\forall u \in V\colon ((u, v_1)) = (u, f)_{L_2} + \varkappa(u, h). \tag{5}$$

Then

$$F(u_0) \geqslant -((v_1 - w, v_1 - w)). \tag{6}$$

Proof. From (4) we get

$$((u, u)) - 2(u, f)_{L_2} - 2\varkappa(u, h) + 2((u, w)) \geqslant -((v, v)) + 2((u, v)) - \\ - 2(u, f)_{L_2} - 2\varkappa(u, h) + 2((u, w)).$$

Then

$$\forall u \in V, \ \forall v \in W_2^k(\Omega)\colon F(u) \geqslant -((v, v)) + 2((u, v)) - 2(u, f)_{L_2} - \\ - 2\varkappa(u, h) + 2((u, w)). \tag{7}$$

Denote $v_1 = v + w$. Then from (7) we get

$$\forall u \in V, \ \forall v_1 \in W_2^k(\Omega)\colon F(u) \geqslant -((v_1 - w, v_1 - w)) - 2(u, f)_{L_2} - 2\varkappa(u, h) + \\ + 2((u, v_1)).$$

From which by use of (5) we have

$$\forall u \in V\colon F(u) \geqslant -((v_1 - w, v_1 - w)).$$

If we have information on regularity of solution of the starting boundary value problem, the construction of the lower estimation may be simplified.

Consider a linear boundary value problem

$$\left.\begin{aligned} Au &= f \quad \text{in } \Omega, \\ B_i u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{8}$$

Denote

$$K = \{u | u \in C^{(2k-1)}(\bar{\Omega}) \cap C^{(2k)}(\Omega),\ Au \in L_2(\Omega)\}.$$

Then

$$D_A = \{u | u \in K,\ u \text{ fulfils all the boundary conditions}\}.$$

Suppose that A, f, Ω and the boundary conditions in (8) are such that the solution u_0 of the problem (8) belongs to D_A. Let $((,))_1$ be a symmetrical bilinear form such that

$$\forall\, u \in K\colon ((u, u))_1 \geqslant 0, \tag{9}$$

$$\forall\, u \in K,\ \forall\, v \in D_A\colon ((u, v))_1 = (Au, v)_{L_2}. \tag{10}$$

Remark. It can be shown that such a form exists for the most of the boundary value problem with Laplace and also with biharmonic operator.

Lemma 1. u_0 minimizes in D_A the functional

$$F_1(u) = ((u, u))_1 - 2(f, u)_{L_2}.$$

Proof. From (9), (10) for $u \in D_A$ we have

$$0 \leqslant ((u - u_0, u - u_0))_1 = ((u, u))_1 - 2(f, u)_{L_2} + ((u_0, u_0))_1. \tag{11}$$

But

$$F_1(u_0) = ((u_0, u_0))_1 - 2(Au_0, u_0)_{L_2} = -((u_0, u_0))_1.$$

From (11) then

$$\forall\, u \in D_A\colon F_1(u) \geqslant F_1(u_0).$$

Theorem 2. Let $v \in K$ be such that $Av = f$. Then

$$F_1(u_0) \geqslant -((v, v))_1 \tag{12}$$

holds.

Proof. $\forall\, u \in D_A$, $\forall\, v \in K$: $((v - u, v - u))_1 \geqslant 0$. Then

$$((u, u))_1 - 2(u, f)_{L_2} \geqslant -((v, v))_1 - 2(u, f)_{L_2} + 2((u, v))_1 .$$

From it follows that

$$\forall\, u \in D_A,\ \forall\, v \in K\colon F_1(u) \geqslant -((v, v))_1 - 2(Av - f, u)_{L_2},$$

which ends the proof.

Example. Consider the boundary value problem

$$-\Delta u = f \quad \text{in } \Omega, \tag{13a}$$

$$u\Big|_{\partial\Omega_1} = \left(\frac{\partial u}{\partial n} + \beta u\right)\Big|_{\partial\Omega_2} = 0, \tag{13b}$$

where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, $\beta = \text{const.} > 0$.

In this case $k = 1$, $r = 2$, $\mu_p = 0$, $t_{11} = 0$, $t_{21} = 0$, $h_{11} = 0$, $h_{21} = 0$, $w = 0$, $V = \{u | u \in W_2^1(\Omega),\ u|_{\partial\Omega_1} = 0$ in the sense of traces$\}$,

$$((v, u_0)) = \int_\Omega \sum_{i=1}^m \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_i}\, \mathrm{d}\Omega + \beta \int_{\partial\Omega_2} u_0 v\, \mathrm{d}S.$$

Assumption (4) is clearly satisfied.

If Ω is such that the solution u_0 of the problem (13) is from $D_{-\Delta}$ we choose the bilinear form

$$((u, v))_1 = \int_\Omega \sum_{i=1}^m u'_{x_i} v'_{x_i}\, \mathrm{d}\Omega + \frac{1}{\beta} \int_{\partial\Omega_2} u'_n v'_n\, \mathrm{d}S, \qquad u, v \in K.$$

Then

$$\forall\, u \in D_{-\Delta}\colon F_1(u) \geqslant -((v, v))_1, \tag{14}$$

where

$$v \in K\colon -\Delta v = f.$$

3. Improvement of the lower estimation

Usually it is not difficult to construct the estimation (12) and sometimes also (6). For the boundary value problem (13) we get the estimation (12) if we find $v \in K$ such that (13a) holds. The estimation (6) we get if e.g. we find $v \in K$ such that (13a) holds and

$$(v'_n + \beta v)|_{\partial\Omega_2} = 0.$$

Denote

$$U = \{v \in W_2^k(\Omega) |\ \forall\, u \in V\colon ((u, v)) = (u, f)_{L_2} + \varkappa(u, h)\}.$$

Lemma 2. The set U is convex and closed.

Proof. Let $v_1 \in U$, $v_2 \in U$ and $0 \leqslant \gamma \leqslant 1$. Then

$$\forall\, u \in V\colon ((u, \gamma v_1)) = (u, \gamma f)_{L_2} + \varkappa(u, \gamma h),$$

$$\forall\, u \in V\colon ((u, (1-\gamma) v_2)) = (u, (1-\gamma) f)_{L_2} + \varkappa(u, (1-\gamma) h).$$

From that

$$((u, v_1 + (1 - \gamma)v_2)) = (u, f)_{L_2} + \varkappa(u, h).$$

Let further $\{v_n\} \subset U$ be fundamental sequence in U. Thus

$$v_n \to v \in W_2^k(\Omega).$$

But

$$\forall n, \ \forall u \in V: ((u, v_n)) = (u, f)_{L_2} + \varkappa(u, h), \tag{15}$$

by this from the continuity of bilinear form in (15) we have

$$\forall u \in V: ((u, v)) = (u, f)_{L_2} + \varkappa(u, h).$$

Denote

$$J(v) = ((v - w, v - w)),$$

where w is a fixed element from $W_2^k(\Omega)$.

Lemma 3. The functional $J(v)$ is convex on $W_2^k(\Omega)$.

Proof. $J(v)$ is differentiable in Gâteaux sense. Then the proof follows from the relation between differentiability and convexity [2].

Lemma 4. The functional $J(v)$ obtains on U the minimum in the element $u_0 + w$.

Proof. As u_0 fulfils (1), there is

$$\forall u \in V: ((u, u_0 + w)) = (u, f)_{L_2} + \varkappa(u, h) \Rightarrow u_0 + w \in U.$$

From (6) we have

$$\forall v \in U: J(v) \geqslant ((u_0, u_0)).$$

At the same time u_0 is the only minimum point of $J(v)$ on U if $((u, u)) = 0 \equiv u = 0$.

Corollary. Relations

$$\forall v \in U: J(u) \leqslant J(v), \tag{16}$$

$$\forall v \in U: J'(u, v - u) \geqslant 0$$

are equivalent.

For the solution of problems (16) it is possible to use methods of variational inequalities.

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SÚHRN

O JEDNEJ METÓDE APOSTERIORNÉHO ODHADU

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Skúma sa okrajová úloha, ktorej slabé riešenie u_0 je definované vzťahom (1). Pri predpokladoch (2), (3) je konštruovaný dolný odhad minima $F(u_0)$. Je ukázané, že na riešenie problému dolného odhadu možno použiť tiež metodiky variačných nerovníc. Tým sa dostávajú ďalšie protismerné (ústretové) metódy k energetickej metóde riešenia (1).

РЕЗЮМЕ

ОБ ОДНОМ МЕТОДЕ АПОСТЕРИОРНОЙ ОЦЕНКИ

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Исследуется краевая задача, обобщенное решение которой определено соотношением (1). При предположениях (2), (3) конструируется нижная оценка для минимума $F(u_0)$. Показано, что для решения задачи о нижней оценке можно тоже применить методики вариационных неравенств. Таким образом получаются встречные методы к энергетическому методу решения (1).