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ON THE CONSTRUCTION OF AN INVARIANT VOLUME

MARTIN KALINA, Bratislava

In his paper [1] J. Kalas constructed a measure, which is invariant with respect to a given group of autohomeomorphisms. In this paper we present similar result using methods of the nonstandard analysis.

According to M. Davis we denote by $*E$ the nonstandard enlargement of a set E and by ${}^0x$ the standard part of a finite hyperreal number x ($x - {}^0x$ is an infinitesimal).

Let (X, T) be a topological space, $(U_n)_{n=1}^{\infty}$ a sequence of open coverings of the space X, H be a group of autohomeomorphisms of X and let $\mathscr{F}$ be a class of closed sets, containing $\emptyset$, which is closed under finite unionns and under the group H. Denote

$$V_1 = U_1,\ V_n = \{E \cap F | E \in V_{n-1},\ F \in U_n\} \qquad n = 2, 3, \ldots$$

Let the following conditions hold:

1) For any $F \in \mathscr{F}$ and any $V \in (V_n)_{n=1}^{\infty}$ there exist $G_i \in V$ and $h_i \in H$ ($i = 1, 2, \ldots, n$) such that $\bigcup_{i=1}^{n} h_i(G_i) \supset F$.

2) For every set $F \in \mathscr{F}$ and for every open nonempty set G such that $F \subset G$ there exists $n \in N$ such that for each $E \in V_n$ and each $h \in H$ there holds

$$\text{if } h(E) \cap F \neq \emptyset, \text{ then } h(E) \subset G.$$

3) Let us suppose that there exists a set $A \in \mathscr{F}$ such that for every set $F \in \mathscr{F}$ there exist $h_i \in H$ ($i = 1, 2, \ldots, n$) such that $\bigcup_{i=1}^{n} h_i(A) \supset F$.

Let $F \in \mathscr{F}$ and $V \in (V_i)_{i=1}$. We define

$$F : V = \min \left\{ n |\ E_i \in V,\ h_i \in H\ (i = 1, 2, \ldots, n) \text{ such that } \bigcup_{i=1}^{n} h_i(E_i) \supset F \right\}$$

$$F : A = \min \left\{ n |\ h_i \in H\ (i = 1, 2, \ldots, n) \text{ such that } \bigcup_{i=1}^{n} h_i(A) \supset F \right\}.$$

Let us define functions λ_i: $\mathscr{F} \to R$ by the formula

$$\lambda_i(E) = \begin{cases} \dfrac{E:V_i}{A:V_i}, & \text{if } E \neq \emptyset \\ 0, & \text{if } E = \emptyset \end{cases}$$

Lemma 1. λ_i are nonnegative, nondecreasing, subadditive functions, which are invariant with respect to the group H.

Proof. We shall prove that the functions λ_i are invariant with respect to H. The rest is clear.

Let $E \in \mathscr{F}$, $g \in H$ and $E: V_i = n$. There exist $G_j \in V_i$ and $h_j \in H$ ($j = 1, 2, \ldots, n$) such that

$$\bigcup_{j=1}^{n} h_j(G_j) \supset E.$$

Evidently

$$g(E) \subset g\left(\bigcup_{j=1}^{n} h_j(G_j)\right) = \bigcup_{j=1}^{n} g \circ h_j(G_j).$$

This implies

$$E: V_i \geqq g(E): V_i \tag{1}$$

Formula (1) implies

$$E: V_i = g^{-1} \circ g(E): V_i \leqq g(E): V_i \leqq E: V_i.$$

Q.E.D.

Lemma 2. For every set $E \in \mathscr{F}$ there holds $\lambda_i(E) \leqq E: A$.

Proof. Let $E \neq \emptyset$. Then $\lambda_i(E) = \dfrac{E:V_i}{A:V_i}$. Let $A: V_i = n$ and $E: A = m$. Then there exist $G_j \in V_i$, $g_j \in H$ ($j = 1, 2, \ldots, n$) and $h_k \in H$ ($k = 1, 2, \ldots, m$) such that

$$\bigcup_{j=1}^{n} g_j(G_j) \supset A, \quad \bigcup_{k=1}^{m} h_k(A) \supset E.$$

This implies

$$\bigcup_{k=1}^{m} h_k\left(\bigcup_{j=1}^{n} g_j(G_j)\right) = \bigcup_{k=1}^{m} \bigcup_{j=1}^{n} h_k \circ g_j(G_j) \supset E.$$

Hence

$$E: V_i \leqq m \cdot n = (E: A) \cdot (A: V_i).$$

Q.E.D.

$(\lambda_i)_{i=1}^{\infty}$ is a sequence of functions λ_i: $\mathscr{F} \to R$, $i \in N$. Hence for $n \in {}^*N \backslash N$ we get a function ${}^*\lambda_n$: ${}^*\mathscr{F} \to {}^*R$.

Lemma 3. Let $n \in {}^*N \backslash N$. Then for any disjoint sets E, F

$${}^*\lambda_n({}^*E) + {}^*\lambda_n({}^*F) = {}^*\lambda_n({}^*(E \cup F)).$$

Proof. Let $E \cap F = \emptyset$. The set $X \backslash F$ is open and $E \subset X \backslash F$. The condition

2 implies that there exists $m \in N$ such that for each $G \in V_m$ and $h \in H$ there holds

$$\text{if } h(G) \cap E \neq \emptyset, \text{ then } h(G) \cap F = \emptyset, \tag{2}$$

$$\text{if } h(G) \cap F \neq \emptyset, \text{ then } h(G) \cap E = \emptyset. \tag{3}$$

By the construction of V_n we get that (2) and (3) hold for each $n > m$ hence also

$$E : V_n + F : V_n = (E \cup F) : V_n$$

for $n > m$. The definition of λ_i implies that

$$\lambda_i(E) + \lambda_i(F) = \lambda_i(E \cup F)$$

for each $i > m$. Therefore

$$*\lambda_n(*E) + *\lambda_n(*F) = *\lambda_n(*(E \cup F))$$

for every $n \in *N \backslash N$. Q.E.D.

For a fixed hyperinteger $n \in *N \backslash N$ define $\lambda: \mathscr{F} \to R$ by the formula $\lambda(E) = {}^{0}*\lambda_n(*E)$.

Theorem. The function λ is a finite volume, which is invariant with respect to the group H.

Proof. This theorem is a corollary of Lemmas 1—3 and the transfer principle.

REFERENCES

[1] Kalas, J.: The construction of an invariant measure, Acta Math. Univ. Comen., 44—45 (1984), 185—194.

[2] Davis, M.: Applied Nonstandard Analysis. New York 1977.

[3] Halmos, P. R.: Measure Theory. New York 1950.

[4] Cutland, N. J.: Nonstandard measure theory and its applications, Bull. London Math. Soc., 15 (1983), 529—589.

SÚHRN

KONŠTRUKCIA INVARIANTNÉHO OBJEMU

Martin Kalina, Bratislava

Cieľom článku bolo podať nový dôkaz existencie objemu, invariantného vzhľadom na danú grupu homeomorfizmov topologického priestoru. Tento dôkaz využíva metódy neštandardnej analýzy.

РЕЗЮМЕ

КОНСТРУКЦИЯ ИНВАРИАНТНОГО ОБЪЕМА

Мартин Калина, Братислава

В стати новое доказательство существования объема, инвариантного ввиду группы гомеоморфизмов данного топологического пространства. Доказательство использует методы нестандартного анализа.