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A NOTE ON THE SHIFT OF THE CIRCLE

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In the paper we present a new proof of the strong ergodicity of the shift of the circle (with an irrational amplitude). (See e.g. [2], theorem 2, ch. 3, § 1.)

We shall represent the circle as an interval $G = \langle 0, 1)$ and the shift T_a as a mapping $T_a: G \to G$, $T_a(x) = x + a \pmod 1$, i.e. $T_a(x) = x + a$, if $x < 1 - a$, or $T_a(x) = x + a - 1$, if $x \geqq 1 - a$, respectively. A measure μ defined on the σ-algebra B of all Borel subsets of G is called to be T_a-invariant, if $\mu(T_a^{-1}(E)) = \mu(E)$ for every $E \in B$.

Theorem. If a is an irrational number, then $T_a: G \to G$ is strongly ergodic, i.e. there is exactly one T_a-invariant probability measure on B.

Proof. Evidently, the Lebesgue measure λ is T_a-invariant. Let μ be another T_a-invariant measure.

Our proof is based on two facts:

1. If a is irrational and $x \in G$, then the trajectory $(T_a^n(x))_n$ is a dense subset of G.

Since μ is also T_a^n-invariant (i.e. T_b-invariant, where $b = na \pmod 1$), we conclude: There exists a dense subset $M \subset G$ such that μ is T_b-invariant for every $b \in M$. (Namely, $M = \{na \pmod 1;\ n \in N\}$).

2. μ is a Haar measure on G, i.e. $\mu(E) = \mu(E + b)$ for every $E \in B$. Of course, it is sufficient to prove the equality for the sets $E = \langle c, d)$, where $0 \leqq c < d < 1$. Since M is dense, there are $a_n \in M$ such that $a_n \searrow b$. Then

$$\mu(\langle c - b, d - b\rangle) = \mu\left(\bigcap_{n=1}^{\infty} \langle c - a_n, d - b\rangle\right) =$$
$$= \lim_{n \to \infty} \mu(\langle c - a_n, d - b + a_n - a_n\rangle) = \lim_{n \to \infty} \mu(\langle c, d - b + a_n\rangle) =$$
$$= \mu\left(\bigcap_{n=1}^{\infty} \langle c, d - b + a_n\rangle\right) = \mu(\langle c, d\rangle).$$

Moreover, $\mu(\{x\}) = 0$ for every $x \in G$. Namely, $T_a^{-n}(\{x\})$ are pairwise disjoint, since a is irrational. Therefore

$$1 \geqq \mu\left(\bigcup_n T_a^{-n}(\{x\})\right) = \sum_n \mu(T_a^{-n}(\{x\})) = \sum_n \mu(\{x\}).$$

Hence we have proved that every T_a-invariant measure μ is a Haar measure. Since G is a compact topological group, there is exactly one Haar probability measure on B (see e.g. [1], theorem C, § 60), i.e. $\mu = \lambda$. So there is exactly one T_a-invariant probability measure on B, i.e. T_a is strongly ergodic.

REFERENCES

[1] Halmos, P. R.: Measure Theory. New York 1950.
[2] Kornfeld, I. P.—Sinaj, Ja. G.-Fomin, S. V.: Ergodic Theory. Moskva 1980.

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SÚHRN

POZNÁMKA O POSUNUTÍ NA KRUŽNICI

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V práci je uvedený nový dôkaz silnej ergodičnosti posunutia (s iracionálnou amplitúdou) na kružnici.

РЕЗЮМЕ

ЗАМЕЧАНИЕ О СДВИГЕ НА ОКРУЖНОСТИ

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В статье приводится новое доказательство строгой эргодичности сдвига (с иррациональной амплитудой) на окружности.