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CERTAIN TYPES OF TRANSFORMATIONS OF MEASURABLE SETS

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Introduction

Let E_n $(n = 1, 2, \ldots)$ denote the n-dimensional Euclidean space, let $\mathscr{L}$ be the family of all Lebesgue measurable subsets of E_n. Given $A \in \mathscr{L}$, we shall denote by $|A|$ the Lebesgue measure of A.

The following transformations of the type T_ω are studied in [1].

Let Ω be a metric space. Suppose that a transformation $T_\omega\colon \mathscr{L} \to \mathscr{L}$ is assigned to each $\omega \in \Omega$. Let the following conditions be satisfied.

(i) There exists $\omega_0 \in \Omega$ such that for every interval $\langle a, b \rangle \subset E_1$ and for every sequence $\{\omega_n\}_{n=1}^{\infty}$ in Ω converging to ω_0 we have

$$\lim_{n \to \infty} (\inf T_{\omega_n}(\langle a, b \rangle)) = a; \quad \lim_{n \to \infty} (\sup T_{\omega_n}(\langle a, b \rangle)) = b.$$

(ii) If E, F are in $\mathscr{L}$ and $E \subset F$, then $T_\omega(E) \subset T_\omega(F)$ for every $\omega \in \Omega$.

(iii) If $E \in \mathscr{L}$ and the sequence $\{\omega_n\}_{n=1}^{\infty}$ converges to ω_0 in Ω, then

$$\lim_{n \to \infty} |T_{\omega_n}(E)| = |T_{\omega_0}(E)| = |E|.$$

The following theorem is true (cf. [1]).

Theorem A. Let Ω and T_ω $(\omega \in \Omega)$ be defined as above and let (i), (ii) and (iii) be satisfied. Let $A \in \mathscr{L}$, $|A| > 0$ and let the sequence $\{\omega_n\}_{n=1}^{\infty}$ converge to ω_0 in Ω. Then there exists a natural number n_0 such that for every $n > n_0$ we have $A \cap T_{\omega_n}(A) \neq \emptyset$.

In [2], the above results are extended from E_1 to any n-dimensional Euclidean space E_n $(n = 1, 2, \ldots)$.

By $S[c, r]$ $(S(c, r))$, we shall denote the closed (open) ball in E_n with centre c and radius r. For every $x \in E_n$, let $\|x\|$ denote the usual norm of x in E_n. If $a \in E_n$, $M \subset E_n$, then $a - M = \{a - x;\ x \in M\}$.

Let Ω be a metric space. Assume that for every $\omega \in \Omega$ there exists a transformation T_ω which transforms measurable sets in E_n to measurable sets in E_n. Let T_ω satisfy the following condicions:

(I) There exists $\omega_0 \in \Omega$ such that for each ball $K = S[a, r] \subset E_n$ and every sequence $\{\omega_n\}_{n=1}^{\infty}$ converging to ω_0 in Ω we have

$$\lim_{n \to \infty} [\sup \{\|y\|;\, y \in a - T_{\omega_n}(K)\}] = r.$$

(II) If $E \subset F$ are measurable subsets of E_n, then $T_\omega(E) \subset T_\omega(F)$ for every $\omega \in \Omega$.

(III) If E is a measurable subset of E_n and sequence $\{\omega_n\}_{n=1}^{\infty}$ converges to ω_0 in Ω, then

$$\lim_{n \to \infty} |T_{\omega_n}(E)| = |T_{\omega_0}(E)| = |E|.$$

The following proposition is proved in [2] for transformations with the properties given above.

Theorem B. Let a sequence $\{\omega_n\}_{n=1}^{\infty}$ converge to ω_0 in Ω. Let T_ω be transformations meeting the conditions (I), (II), and (III). Let A be a set having positive measure in E_n. Then there is a natural number n_0 such that, for any $n > n_0$, the set $A \cap T_{\omega_n}(A)$ has positive measure.

Interrelation between properties (I)—(III) and (i)—(iii)

It is natural to ask how in the space E_1, the properties (I)—(III) from [2] are related to the properties (i)—(iii) from [1].

Theorem 1. In the space E_1, the conditions (i)—(iii) are equivalent with (I)—(III).

Proof. It is immediately seen that (II) and (III) are the same as (ii) and (iii). It is therefore sufficient to check the relation between (I) and (i).

Let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence converging to ω_0. Then, assuming (i) to be true, for every interval $\langle a, b \rangle$ and every $\varepsilon > 0$ there is n_0 such that for every $n > n_0$ we have

$$\inf T_{\omega_n}(\langle a, b \rangle) \in (a - \varepsilon;\, a + \varepsilon),$$
$$\sup T_{\omega_n}(\langle a, b \rangle) \in (b - \varepsilon;\, b + \varepsilon).$$

Therefore, if $K = S[s, r] = \langle s - r, s + r \rangle$ is any ball, it follows that whenever $\varepsilon > 0$, there exists n_0 such that for every $n > n_0$ we have

$$\inf T_{\omega_n}(K) \in (s - r - \varepsilon,\, s - r + \varepsilon),$$
$$\sup T_{\omega_n}(K) \in (s + r - \varepsilon,\, s + r + \varepsilon).$$

Hence $\sup \|s - T_{\omega_n}(K)\| \in (r - \varepsilon, r + \varepsilon)$. Honewer, since the last relation is true for any $\varepsilon > 0$, we infer that

$$\lim_{n \to \infty} \{\sup \|s - T_{\omega_n}(K)\|\} = r,$$

for every sequence $\{\omega_n\}_{n=1}^{\infty}$ converging to ω_0. Thus (i) implies (I).

On the other hand, we can see that (I) does not imply (i). In fact, if $K = S[s, r]$ is any ball, it is sufficient to choose $T_\omega(K) = \langle s - r, s\rangle$ for all $\omega \in \Omega$ to meet (I) but not (i). However, we are going to prove that (i) follows from the conditions (I) and (III).

Assume, therefore, conditions (I) and (III). Let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence converging to ω_0 and let $\langle a, b\rangle$ $(a < b)$ be any interval. Denoting its length by $2r$ and its centre by s, we put $\langle a, b\rangle = S[s, r] = K$. Now suppose, on the contrary, that (i) does not hold. Then at least one of the equalities

$$\lim_{n \to \infty} \{\inf T_{\omega_n}(\langle a, b\rangle)\} = a, \quad \lim_{n \to \infty} \{\sup T_{\omega_n}(\langle a, b\rangle)\} = b$$

is false, i.e. the sequence $\{\inf T_{\omega_n}(\langle a, b\rangle)\}_{n=1}^{\infty}$ has a limit point different from a or the sequence $\{\sup T_{\omega_n}(\langle a, b\rangle)\}_{n=1}^{\infty}$ has a limit point distinct from b. If there were a limit point of $\{\inf T_{\omega_n}(\langle a, b\rangle)\}_{n=1}^{\infty}$ less than a, or a limit point of $\{\sup T_{\omega_n}(\langle a, b\rangle)\}_{n=1}^{\infty}$ greater than b, we would immedately obtain a contradiction with (I). Hence we infer that all limit points of the two sequences are in $\langle a, b\rangle$. We show that none of them can be in the open interval (a, b).

To be specific, suppose that some $c \in (a, b)$ is a limit point of

$$\{\inf T_{\omega_n}(\langle a, b\rangle)\}_{n=1}^{\infty}.$$

Then there is a subsequence $\{\inf T_{\omega_{n_i}}(\langle a, b\rangle)\}_{n=1}^{\infty}$ converging to $c > a$. In that case however, for any ε with $0 < \varepsilon < \dfrac{c - a}{4}$ there exists n_{i0} such that for all $n_i > n_{i0}$ we have $T_{\omega_{n_i}}(K) \subset c - \varepsilon, b - \varepsilon)$. Therefore

$$|T_{\omega_{n_i}}(K)| \leqq b - c + 2\varepsilon < b - c + \frac{e - a}{2} = b - \frac{a + c}{2},$$

and hence

$$\lim_{n \to \infty} |T_{\omega_{n_i}}(\langle a, b\rangle)| \leqq b - \frac{a + c}{2} < b - a,$$

which contradicts (III).

The theorem is proved.

In view of the last theorem we can say that Theorem B is a generalization of and an improvement upon Theorem A.

Transformations of a similar type in topological spaces

Let X be an arbitrary nonempty set. Let $\mathscr{C}$ and $\mathscr{U}$ be families of subsets of X. Let $\mathscr{S}$ denote a σ-ring of subsets of X with $\mathscr{C} \subset \mathscr{S}$ and $\mathscr{U} \subset \mathscr{S}$. Let μ be a measure on $\mathscr{S}$.

Let Ω be a metric space and let for each $\omega \in \Omega$ there exist a transformation $T_\omega \colon \mathscr{S} \to \mathscr{S}$ with the following properties:

(a) There exists $\omega_0 \in \Omega$ such that for all sequences $\{\omega_n\}_{n=1}^\infty$ converging to ω_0 we have: Whenever $F, G \subset X$, $F \in \mathscr{C}$, $G \in \mathscr{U}$, $F \subset G$ then there is n_0 such that $T_{\omega_n}(F) \subset G$ for every $n > n_0$.

(b) If $E, F \in \mathscr{S}$, $E \subset F$, then $T_\omega(E) \subset T_\omega(F)$ for all $\omega \in \Omega$.

(c) If a sequence $\{\omega_n\}_{n=1}^\infty$ converges to ω_0, then for all $E \in \mathscr{S}$ we have

$$\lim_{n \to \infty} \mu(T_{\omega_n}(E)) = \mu(T_{\omega_0}(E)) = \mu(E).$$

Definition 1. a) Let X, $\mathscr{C}$, $\mathscr{U}$ and $\mathscr{S}$ have the same meaning as above. We shall say that the measure μ is $\mathscr{C} - \mathscr{U}$-regular if for every $E \in \mathscr{S}$

$$\mu(E) = \sup \{\mu(C),\ C \in \mathscr{C},\ C \subset E\} = \inf \{\mu(U),\ U \in \mathscr{U},\ E \subset U\}.$$

b) Let $(X, \mathscr{U})$ be a Hausdorff topological space. Denote by $\mathscr{C}$ the family of all compact subsets of X. Let $\mathscr{S}$ be a σ-algebra containing all open sets, i.e. $\mathscr{U} \subset \mathscr{S}$. Let μ be a measure on $\mathscr{S}$. We shall say that μ is regular if it is $\mathscr{C} - \mathscr{U}$-regular.

Theorem 2. Let Ω be a metric space and X an arbitrary nonempty set. Let $\mathscr{C}$, $\mathscr{U}$ and $\mathscr{S}$ be families of subsets of X, such that $\mathscr{S}$ is a σ-ring and $\mathscr{C}, \mathscr{U} \subset \mathscr{S}$. Let μ be a $\mathscr{C} - \mathscr{U}$-regular measure on $\mathscr{S}$. Let for each $\omega \in \Omega$ there exist a transformation $T_\omega \colon \mathscr{S} \to \mathscr{S}$ with the properties (a), (b) and (c). Then the following is true.

If $E \in \mathscr{S}$, $\mu(E) = \alpha$, $0 < \alpha < +\infty$, γ is a number in the interval $(0, \alpha)$ and $\{\omega_n\}_{n=1}^\infty$ is any sequence in Ω converging to ω_0, then there is such an index n_0 that $\mu(E \cap T_{\omega_n}(E)) > \gamma$ for every $n > n_0$.

Proof. Let $E \in \mathscr{S}$, $\mu(E) = \alpha$, $0 < \alpha < +\infty$. Let a sequence $\{\omega_n\}_{n=1}^\infty$ converge to ω_0 in the space Ω. Choose $\gamma \in (0, \alpha)$. Due to the $\mathscr{C} - \mathscr{U}$-regularity of μ there exists a set $F \in \mathscr{C}$ such that $F \subset E$ and

$$\mu(F) > \gamma + \frac{3}{4}(\alpha - \gamma) = \frac{3\alpha + \gamma}{4}.$$

On the other hand, the $\mathscr{C} - \mathscr{U}$-regularity of μ also implies that there exists a set $G \in \mathscr{U}$ with $E \subset G$ and $\mu(G - E) < \dfrac{\alpha - \gamma}{4}$.

Since $F \in \mathscr{S}$, $G \in \mathscr{U}$ and $F \subset G$, by (a) we infer that there is n' such that for every $n > n'$ we have

$$T_{\omega_n}(F) \subset G. \tag{1}$$

However, due to the condition (c) there exists n'' such that $n > n''$ implies

$$\mu(T_{\omega_n}(F)) > \gamma + \frac{1}{2}(\alpha - \gamma) = \frac{\alpha + \gamma}{2}. \tag{2}$$

Denote $n_0 = \max\{n', n''\}$. Now, for every $n > n_0$, both (1) and (2) will be true and in view of (b) we can write

$$\begin{aligned} \frac{\alpha+\gamma}{2} &< \mu(T_{\omega_n}(F)) = \mu((T_{\omega_n}(F)) \cap G) = \\ &= \mu((E \cup (G - E)) \cap T_{\omega_n}(F)) = \\ &= \mu(E \cap T_{\omega_n}(F)) + \mu((G - E) \cap T_{\omega_n}(F)) \leqq \\ &\leqq \mu(E \cap T_{\omega_n}(E)) + \mu((G - E) \cap T_{\omega_n}(F)) \leqq \\ &\leqq \mu(E \cap T_{\omega_n}(E)) + \frac{\alpha - \gamma}{4}. \end{aligned}$$

Hence for all $n > n_0$ we get

$$\mu(E \cap T_{\omega_n}(E)) > \frac{\alpha - \gamma}{2} - \frac{\alpha - \gamma}{4} = \frac{\alpha + 3\gamma}{4} > \gamma.$$

The theorem is proved.

Remark 1. The above proof shows that Theorem 2 will be true also if we consider, instead of the measure μ on the σ-ring $\mathscr{S}$, any monotone subadditive set function defined on a ring containing the families $\mathscr{C}$ and $\mathscr{U}$.

As we have already mentioned, further properties of transformations T_ω will be studied in topological spaces.

Corollary 1. Let Ω be a metric space and $(X, \mathscr{U})$ a topological space. Let $\mathscr{S}$ be a σ-algebra of subsets of X, containing all open and compact subsets. Let μ be a regular measure defined on $\mathscr{S}$. Let for every $\omega \in \Omega$ there exist a transformation T_ω: $\mathscr{S} \to \mathscr{S}$. Let the transformations T_ω satisfy (a), (b) and (c). (Here, $\mathscr{C}$ denotes the family of all compact subsets of X.) Then for any $E \in \mathscr{S}$, $\mu(E) = \alpha$, $0 < \alpha < +\infty$, $\gamma \in (0, \alpha)$ and any sequence $\{\omega_n\}_{n=1}^{\infty}$ of elements of Ω converging to ω_0 there exists n_0 such that

$$\mu(E \cap T_{\omega_n}(E)) > \gamma.$$

for all $n > n_0$.

Proof. Quite analogous to that of Theorem 2. It is sufficient to let $\mathscr{C}$ be the family of all compact subsets of X. Since by hypothesis, $\mathscr{S}$ is a σ-algebra containing both the family $\mathscr{U}$ of all open sets and the family $\mathscr{C}$ of all compact sets, regularity of the measure in that case coincides with its $\mathscr{C} - \mathscr{U}$-regularity in the sense of Definition 1.

Now we can observe the connection between properties (a)—(c) and (I)—(III) in Euclidean spaces.

Definition 2. We shall say that a transformation T_ω has the property (*) if for any two closed balls $K_1 = S[a_1, r_1]$ and $K_2 = S[a_2, r_2]$ in E_n we have

$$T_\omega(K_1 \cup K_2) = T_\omega(K_1) \cup T_\omega(K_2).$$

Theorem 3. Let E_n ($n = 1, 2, \ldots$) be the n-dimensional Euclidean space with Lebesgue measure μ. Denote by $\mathscr{C}$ the family of all compacts and by $\mathscr{U}$ the family of all open sets in E_n. Then

a) The assumptions (a), (b), (c) imply the properties (I), (II), (III).

b) If the transformations T_ω satisfy (*) for all $\omega \in \Omega$, then (I)—(III) and (a)—(c) are equivalent.

Proof.

a) Assumptions (b) and (c) are the same as (II) and (III). If the transformations T_ω satisfy (a), they still need not meet (I). For example, put $T_\omega(S[a, r]) = S\left[a, \frac{r}{2}\right]$ for each $\omega \in \Omega$ and every ball $S[a, r]$. However, it can be easily deduced from the properties of the usual topology in Euclidean spaces that whenever the transformations T_ω satisfy (a) and (c), then they satisfy also (I).

b) In view of the proof of part a) it is sufficient to show that (I), (II), (III) and (*) imply (a). We are going to show that (a) is implied by (I), (II) and (*). (Later we shall give an example showing that even in E_1 the properties (I), (II), (III) need not imply (a).)

Let $F \subset G$, $F \in \mathscr{C}$, $G \in \mathscr{U}$ are arbitrary sets. Since G is an open subset of E_n, it can be expressed in the form

$$G = \bigcup_{i=1}^{\infty} S(a_i, r_i),$$

where $S(a_i, r_i)$ ($i = 1, 2, \ldots$) are open balls whose closures are subsets of G. These balls cover the compact set F as well. Therefore we can choose finitely many balls with

$$F \subset \bigcup_{j=1}^{m} S(a_{i_j}, r_{i_j}),$$

and also $F \subset \bigcup_{j=1}^{m} S[a_{i_j}, r_{i_j}]$. Put

$$F_j = F \cap S[a_{i_j}, r_{i_j}], \qquad (j = 1, 2, \ldots, m).$$

For all $j = 1, 2, \ldots, m$, F_j is a compact set and moreover

$$F_j \subset S[a_{ij}, r_{ij}] \subset G.$$

In view of (II) for each $\omega \in \Omega$ we have

$$T_\omega(F_j) \subset T_\omega(S[a_{ij}, r_{ij}]). \tag{3}$$

Now let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence converging to ω_0 in Ω. By (I), for every $j = 1, 2, \ldots, m$ there exists n_j such that for all $n < n_j$ we have

$$T_{\omega_n}(S[a_{ij}, r_{ij}])\ G. \tag{4}$$

Put $n_0 = \max\{n_1, n_2, \ldots, n_m\}$. Then due to (3) and (4) we shall have for all $n > n_0$ and for all sets F_j $(j = 1, 2, \ldots, m)$

$$T_{\omega_n}(F_j) \subset G.$$

Since (∗) implies an analogous proposition for any finite number of closed balls, for all $n > n_0$ we get

$$T_{\omega_n}(F) = T_{\omega_n}\left(\bigcup_{j=1}^{m} F_j\right) \subset T_{\omega_n}\left(\bigcup_{j=1}^{m} S[a_{ij}, r_{ij}]\right) = \bigcup_{j=1}^{m} T_{\omega_n}(S[a_{ij}, r_{ij}]) \subset G.$$

The proof is complete.

The following example shows that if the transformations T_ω have the properties (I)—(III) but have not the property (∗), then they need not have the property (a).

Example 1. Let $\Omega \neq \emptyset$ be an arbitrary metric space and ω_0 any point in Ω. For each $\omega \in \Omega$ define $T_\omega\colon 2^{E_1} \to 2^{E_1}$ by the following rule

$$T_\omega(M) = \begin{cases} M \text{ if } M \subset E_1 \text{ and } \{0;\, 2\} \not\subset M \\ M \cup \{1\} \text{ if } M \subset E_1 \text{ and } \{0;\, 2\} \subset M. \end{cases}$$

Conditions (I), (II) and (III) are satisfied but the transformations thus defined fail to have the property (∗). It suffices to choose

$$K_1 = \left\langle -\frac{1}{2}, \frac{1}{2}\right\rangle, \quad K_2 = \left\langle \frac{3}{2}, \frac{5}{2}\right\rangle.$$

Then for every $\omega \in \Omega$ we have

$$T_\omega(K_1 \cup K_2) = \left\langle -\frac{1}{2}, \frac{1}{2}\right\rangle \cup \left\langle \frac{3}{2}, \frac{5}{2}\right\rangle \cup \{1\},$$

but

$$T_\omega(K_1) \cup T_\omega(K_2) = \left\langle -\frac{1}{2}, \frac{1}{2}\right\rangle \cup \left\langle \frac{3}{2}, \frac{5}{2}\right\rangle.$$

In a similar way it is easy to see that the transformations T_ω do not have the property (a).

Remark 2. Denote by $\mathscr{L}$ the family of all Lebesgue measurable subsets of the space E_n. Let the transformations T_ω: $\mathscr{L} \to \mathscr{L}$ be induced by suitable point mappings g_ω: $E_n \to E_n$. Then these transformations are known to have the property (∗), and hence for then the conditions (I)—(III) are equivalent tc (a)—(c).

Remark 3. As shown by the following example, there exist transformaticns of the type T_ω having the property (∗) which are not induced by point mappings.

Example 2. Let $\Omega = E_1$ with the Euclidean metric. If $E \in \mathscr{L}$ and $\omega \in \Omega$ then

$$T_\omega(E) = \begin{matrix} E \text{ if } 0 \notin E \\ E \cup (-\omega, +\omega) \text{ if } 0 \in E. \end{matrix}$$

Evidently for any $E \in \mathscr{L}$ and each $\omega \in \Omega$ we have $T_\omega(E) \in \mathscr{L}$. The transformations just defined have evidently also the property (∗) and if we put $\omega_0 = 0$ they will have properties (I)—(III), too.

In the case of spaces E_n ($n = 1, 2, \ldots$) we can state the following proposition which improves the previous results for certain types of transformations.

Corollary 2. Let T_ω ($\omega \in \Omega$) denote transformations which are induced by suitable point mappings or have the property (∗) and satisfy the conditions (I), (II), (III). Let $A \in \mathscr{L}$, $|A| = \alpha$, $0 < \alpha < +\infty$ and let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence converging to ω_0 in Ω. Then to overy $\gamma \in (0, \alpha)$ there exists N_0 such that for all $n > N_0$ we have

$$|A \cap T_{\omega_n}(A)| > \gamma.$$

Proof. In view of Theorem 3, the hypotheses of Corollary 1 are satisfied and hence our proposition follows immediately.

As can be immediately seen, for transformations induced by point mappings or those enjoying property (∗), our Corollary 2 is stronger than the assertion of Theorem B (and therefore also stronger than Theorem A) in Introduction.

The assertion of Corollary 1 can be strengthened in the following way.

Theorem 4. Let Ω be a metric space and $(X, \mathscr{U})$ a Hausdorff topological space. Let $\mathscr{S}$ be a σ-algebra of subsets of X containing all open sets and let μ be a regular measure on $\mathscr{S}$. Let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence converging to ω_0 in Ω and let T_ω be transformations satisfying (a), (b) and (c). Let $E \in \mathscr{S}$, $\mu(E) = \alpha$, $0 < \alpha < \infty$.

Then for every $\gamma \in (0, \alpha)$ there exists a subsequence $\{\omega_{nk}\}_{k=1}^{\infty}$ of $\{\omega_n\}_{n=1}^{\infty}$ such that

$$\mu\left(\bigcap_{K=1}^{\infty} (E \cap T_{\omega_{nk}}(E))\right) > \gamma.$$

Proof. Let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence converging to $\omega_0 \in \Omega$. Let T_{ω_n} satisfy (a), (b) and (c). Let $E \in \mathcal{S}$ and $0 < \mu(E) = \alpha < +\infty$. Then Corollary 1 there is N_1 such that for all $n > N_1$ we have

$$\mu(E \cap T_{\omega_n}(E)) > \frac{3\alpha + \gamma}{4}, \quad \text{i.e.} \quad \mu(E - T_{\omega_n}(E)) < \frac{\alpha - \gamma}{4}.$$

Now choose ω_{n1} with $n_1 > N_1$.

Similarly, Corollary 1 guarantees the existence of some $N_2 > n_1$ such that for all $n > N_2$ we have

$$\mu(E \cap T_{\omega_n}(E)) > \frac{7\alpha + \gamma}{8} \quad \text{i.e.} \quad \mu(E - T_{\omega_n}(E)) < \frac{\alpha - \gamma}{8}.$$

Choose ω_{n2} such that $n_2 > N_2$.

Suppose that we have already found points $\omega_{n1}, \omega_{n2}, \ldots, \omega_{nk}$ such that for each $i = 1, 2, \ldots, k$ we have

$$\mu(E - T_{\omega_{ni}}(E)) < \frac{\alpha - \gamma}{2^{i+1}}.$$

Then the point ω_{nk+1} can be found as follows.

By Corollary 1 there exists $N_{k+1} > n_k$ such that for all $n > N_{k+1}$ we have

$$\mu(E \cap T_{\omega_n}(E)) > \frac{(2^{n+2} - 1)\alpha + \gamma}{2^{n+2}} \quad \text{i.e.} \quad \mu(E - T_{\omega_n}(E)) < \frac{\alpha - \gamma}{2^{n+2}}.$$

It is sufficient now to choose ω_{nk+1} with $n_{k+1} > N_{k+1}$. In such a way, a subsequence with the claimed properties can be constructed by induction.

Since a σ-algebra is closed under taking countable unions, we may write

$$\mu\left(E - \bigcap_{K=1}^{\infty} (E \cap T_{\omega_{nk}}(E))\right) = \mu\left[\bigcup_{K=1}^{\infty} (E - T_{\omega_{nk}}(E))\right] \leqq \sum_{K=1}^{\infty} \frac{\alpha - \gamma}{2^{k+1}} = \frac{\alpha - \gamma}{2}.$$

From

$$\mu\left(E - \bigcap_{K=1}^{\infty} (E \cap T_{\omega_{nk}}(E))\right) = \mu(E) - \mu\left(\bigcap_{K=1}^{\infty} (E \cap T_{\omega_{nk}}(E))\right) \leqq \frac{\alpha - \gamma}{2}$$

we obtain

$$\mu\left(\bigcap_{K=1}^{\infty} (E \cap T_{\omega_{nk}}(E))\right) \geqq \alpha + \frac{\alpha - \gamma}{2} > \gamma.$$

The theorem is proved.

Corollary 3. In case $X = E_n$, the preceding theorem can be reworded as follows.

Let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence converging to ω_0 in Ω. Let T_{ω_n} be transformations induced by a suitable point mapping or having the property (*). Let T_{ω_n} satisfy (I), (II), (III). Let A be a measurable set with $\mu(A) = \alpha > 0$ ($\alpha < +\infty$). Then for any $\gamma \in (0, \alpha)$ there exists a subsequence such that

$$\mu(A \cap T_{\omega_{n1}}(A) \cap T_{\omega_{n2}}(A) \cap \ldots \cap T_{\omega_{nk}}(A) \cap \ldots) > \gamma.$$

Theorem 5. Suppose the hypotheses of Theorem 4 are fulfilled. Then for every $\gamma \in (0, \alpha)$ there exists a subsequence $\{\omega_{nk}\}_{k=1}^{\infty}$ and a measurable set $A \subset E$ such that $\mu(A) > \gamma$ and $T_{\omega_{nk}}(A) \subset E$ for every ω_{nk} ($k = 1, 2, \ldots$).

Proof. The hypotheses of Theorem 4 being fulfilled, a seubsequence $\{\omega_{nk}\}_{k=1}^{\infty}$ of $\{\omega_n\}_{n=1}^{\infty}$ can be choosen to satisfy the proposition of Theorem 4. Put $A_0 = E$ and $A_k = A_{k-1} \cap T_{\omega_{nk}}(E)$ for $k = 1, 2, \ldots$ Then $A_0 \supset A_1 \supset \ldots \supset A_k \supset \ldots$ and moreover

$$A_k = E \cap \left(\bigcap_{i=1}^{K} T_{\omega_{ni}}(E) \right) = \bigcap_{i=1}^{K} (E \cap T_{\omega_{ni}}(E)).$$

We are going to show that the set $A = \bigcap_{K=1}^{\infty} A_k$ has the claimed properties. A is measurable because it is a countable intersection of measurable sets. Its measure is

$$\mu(A) = \mu\left(\bigcap_{K=1}^{\infty} A_k \right) = \lim_{k \to \infty} \mu(A_k) = \lim_{k \to \infty} \mu\left(\bigcap_{i=1}^{K} (E \cap T_{\omega_{ni}}(E)) \right) =$$
$$= \mu\left(\bigcap_{i=1}^{\infty} (E \cap T_{\omega_{ni}}(E)) \right) > \gamma,$$

which follows from Theorem 4.

Thus the theorem is proved.

In the special case $X = E_n$ we obtain the following corollary to the last theorem.

Corollary 4. Let transformations $T_{\omega}: \mathscr{L} \to \mathscr{L}$ satisfy the assumtions (I), (II), (III) and be induced by point mappings or satisfy (*). Let $\{\omega_n\}_{n=1}^{\infty}$ be any sequence converging to $\omega_0 \in \Omega$ and let $A \in \mathscr{L}$ be a set with $0 < \mu(A) = \alpha < +\infty$. Let $\gamma \in (0, \alpha)$. Then there exists a subsequence $\{\omega_{nk}\}_{k=1}^{\infty}$ and a measurable set $B \subset A$ such that $\mu(B) > \gamma$ and for each $\zeta \in B$ and every ω_{nk} ($k = 1, 2, \ldots$) we have $T_{\omega_{nk}}(\{\zeta\}) \subset A$, i.e. $T_{\omega_{nk}}(B) \subset A$ for every ω_{nk} ($k = 1, 2, \ldots$).

Proof. It follows from Theorem 3 and Remark 2 that the transformations T_{ω} considered in the Corollary satisfy (a), (b) and (c). Therefore the Corollary follows immediately from Theorem 5.

Remark 4. For certain classes of transformations, Corollary 4 contains under weaker assumptions an essentially stronger statement than Theorem 3 of [2].

Theorem 6. Let $(X, \|)$ be a T_1-space. Let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence converging to ω_0 in Ω and set transformations T_{ω_n} satisfy (a). Let $G \subset X$ be an open subset of X. Then for every $f \in G$ there is n_0 such that $T_{\omega_n}(\{g\}) \subset G$ whenever $n > n_0$.

Proof. Let $g \in G$. Then $\{g\} \subset G$ and $\{g\}$ is a closed and compact subset of the open set G. The proposition of the theorem now follows directly from the property (a).

REFERENCES

[1] Neubrunn, T.—Šalát, T.: Distance sets, ratio sets and certain transformations of sets real numbers, Ĉas. pro pěs. Mat. 94, č. 4, (1969), p. 381—393.

[2] Pál, M.: On certain transformations of sets in R_N, AFRNUC — Math. XXIX (1974), p. 43—53.

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SÚHRN

ISTÉ TYPY TRANSFORMÁCIÍ MERATEĽNÝCH MNOŽÍN

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V práci sa študuje súvis istých tried transformácií merateľných množín v jednorozmernom euklidovskom priestore E_1. Zároveň je tu ukázané, že ak uvažujeme o transformáciách podobného typu v topologických priestoroch, dostaneme silnejšie a všeobecnejšie tvrdenia.

РЕЗЮМЕ

НЕКОТОРЫЕ ТИПЫ ТРАНСФОРМАЦИЙ ИЗМЕРЫМЫХ МНОЖЕСТВ

Йарослав Червенанскы, Братислава

В работе изучена связь некоторых классов трансформаций измеримых множеств в одноразмерном Евклидовом пространстве E_1. Здесь также показано, что когда мы рассматрываем трансформации подобного типа в топологических пространствах, можно получить более сильные и более общие утверждения.