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A NOTE ON POINTS OF ABSOLUTE CONTINUITY OF SYMMETRICALLY DIFFERENTIABLE FUNCTIONS

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If f: $[a, b] \to R$, then $p \in [a, b]$ is said to be a point of absolute continuity of f providing there is a $\delta > 0$ such that f is absolutely continuous on the interval $[p - \delta, p + \delta] \cap [a, b]$. This notion was introduced by M. C. Chakrabarty and P. C. Bhakta, and Chakrabarty in [1] and [2], respectively, and the study of such points was continued by T. Šalát in [4]. Of particular interest are the general properties of the set of all points of absolute continuity of the function f: $[a, b] \to R$, denoted $G(f)$, and the related set $N(f) = [a, b] - G(f)$. It is evident that for arbitrary f, $G(f)$ is open in the relative topology of $[a, b]$ and also, therefore, that $N(f)$ is closed. In [4], T. Šalát proved the following theorem concerning symmetrically differentiable functions. The function f is said to be symmetrically diferentiable at a point x if

$$f^{(s)}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists.

Theorem S. Suppose f is continuous, $f^{(s)}(x)$ exists and is finite everywhere in (a, b), and f satisfies property (V). Then the set $N(f)$ is nowhere dense in $[a, b]$.

Here the property (V) is a technical condition relating certain derivatives of f on certain subsets of $[a, b]$. The purpose of this note is to prove that the conclusion of Theorem S holds even if one deletes property (V) from the hypothesis. That is:

Theorem HS. Suppose f is continuous and $f^{(s)}(x)$ exists and is finite everywhere in (a, b). Then the set $N(f)$ is nowhere dense in $[a, b]$.

The proof follows the same lines as the proof of Theorem S but makes use of the following two theorems which can be found in [3]. The first result is due to L. Larson, while the second, often referred to as the Quasi-Mean Value Theorem for Symmetric Derivatives was apparently first proved by C. Aull.

A fairly complete history of these and other symmetric behaviour results can be found in L. Larson's survey, [3]. In the statement of Theorem L, B_1 denotes the set of functions of the first class of Baire.

Theorem L. If the function f is such that $f^{(s)}$ exists everywhere (finite or infinite), then $f^{(s)} \in B_1$.

Theorem QMV. Suppose f is continuous, and $f^{(s)}(x)$ exists and is finite everywhere and $a < b$. Then there are numbers c, $d \in (a, b)$ such that

$$f^{(s)}(c) \leqq \frac{f(b) - f(a)}{b - a} \leqq f^{(s)}(d)$$

The proof of our theorem is as follows:

Proof of Theorem HS. As $f^{(s)} \in B_1$, then the set of points of discontinuity of $f^{(s)}$, denoted $D(f^{(s)})$, is of the first Baire category. If $x \in R - D(f^{(s)})$, then since $f^{(s)}(x)$ is finite, there is a $\delta > 0$ and an M such that $|f^{(s)}(y)| \leqq M$ whenever $|x - y| < \delta$. But then, if $x - \delta < A < B < x + \delta$,

$$\left|\frac{f(B) - f(A)}{B - A}\right| \leqq M$$

and consequently, f is Lipschitzian on $[x - \delta, x + \delta]$. It follows that $N(f) \subseteqq D(f^{(s)})$ and $N(f)$ is closed, the theorem follows.

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РЕЗЮМЕ

ЗАМЕЧАНИЕ О ТОЧКАХ АБСОЛЮТНОЙ НЕПРЕРЫВНОСТИ СИММЕТРИЧЕСКИ ДИФФЕРЕНЦИРУЕМЫХ ФУНКЦИЙ

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Точка $p \in [a, b]$ называется точкой абсолютной непрерывности функции $f\colon [a, b] \to R$, если функция f абсолютно непрерывна на интервале $[p - \delta, p + \delta] \cap [a, b]$ для некоторого $\delta > 0$. Обозначим символом $G(f)$ множество всех точек абсолютной непрерывности функции f. В работе доказано, что множество $[a, b] - G(f)$ является нигде неплотным множеством, если f непрерывная на $[a, b]$ и симметрически дифференцируемая на (a, b).

SÚHRN

POZNÁMKA O BODOCH ABSOLÚTNEJ SPOJITOSTI SYMETRICKY DIFERENCOVATEĽNÝCH FUNKCIÍ

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Bod $p \in [a, b]$ sa nazýva bodom absolútnej spojitosti funkcie $f\colon [a, b] \to R$, ak existuje také $\delta > 0$, že funkcia f je absolútne spojitá na intervale $[p - \delta, p + \delta] \cap [a, b]$. Označme znakom $G(f)$ množinu všetkých bodov absolútnej spojitosti funkcie f. V práci je dokázané, že množina $[a, b] - G(f)$ je riedka v intervale $[a, b]$, ak funkcia f je spojitá na $[a, b]$ a symetricky diferencovateľná na intervale (a, b).