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APPLICATIONS OF THE CATEGORY METHOD IN THE THEORY OF MODULAR SEQUENCE SPACES

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1. Preliminaries

Let X be a real linear space. A functional $\varrho\colon X \to \langle 0, +\infty\rangle$ is said to be a convex modular if

(a) $\varrho(x) = 0$ if and only if $x = 0$;
(b) $\varrho(-x) = \varrho(x)$;
(c) $\varrho(\alpha x + \beta y) \leqq \alpha\varrho(x) + \beta\varrho(y)$ for $x, y \in X$, $\alpha, \beta \geqq 0$, $\alpha + \beta = 1$.

The set

$$X_\varrho = \left\{x \in X\colon \underset{t>0}{\exists}\ \varrho(tx) < +\infty\right\}$$

is a linear subspace of X and it is called the modular space determined by ϱ.

The formula

$$\|x\| = \inf\left\{t > 0\colon \varrho\left(\frac{x}{t}\right) \leqq 1\right\}$$

defines a norm on X_ϱ (cf. [4], [5]).

In what follows we shall use the following auxiliary result.

Lemma 1.1. Let C be a subset of a modular space X_ϱ. If for each $x \in C$ there exists an $\varepsilon > 0$ such that

$$\{y \in X_\varrho\colon \varrho(x - y) < \varepsilon\} \subset C$$

then C is open in X_ϱ.

Proof. Let $x \in C$. According to the assumption there exists such an $\varepsilon > 0$ that

$$W_\varepsilon(x) = \{y \in X_\varrho : \varrho(x-y) < \varepsilon\} \subset C$$

We can assume that $\varepsilon < 1$. It suffices to prove that

$$K(x, \varepsilon) = \{y \in X_\varrho : \|x - y\| < \varepsilon\} \subset C$$

Let $y \in K(x, \varepsilon)$. Then according to the definition of the norm we get from $\|x - y\| < \varepsilon' < \varepsilon$ the inequality

$$\frac{1}{\varepsilon'}\, \varrho(x-y) \leqq \varrho\left(\frac{x-y}{\varepsilon'}\right) \leqq 1$$

Hence $\varrho(x-y) \leqq \varepsilon' < \varepsilon$, $y \in W_\varepsilon(x) \subset C$.

A function $f\colon \langle 0, +\infty) \to \langle 0, +\infty)$ is said to be an Orlicz function if it is continuous, non-decreasing, convex and $\lim_{t\to\infty} f(t) = +\infty$. If $f(t) = 0$ for some $t > 0$, then f is said to be a degenerate Orlicz function (cf. [3], p. 137).

An Orlicz function is said to satisfy the Δ_2-condition for small t if there exists $K > 0$ and $t_0 > 0$ such that $f(2t) \leqq K f(t)$ for each $t \in \langle 0, t_0\rangle$ (cf. [2], [4]).

Let f be a non-degenerate Otlicz function whose right-derivative P satisfies $P(0) = 0$ and $\lim_{t\to\infty} P(t) = +\infty$. The right-inverse Q of P given by $Q(u) = \sup \{t\colon P(t) \leqq u\}$ (for $u \geqq 0$) is a right-continuous non-decreasing function such that $Q(0) = 0$ and $Q(u) > 0$ for $u > 0$. Put $f^*(t) = \int_0^t Q(u)\,du$ for $t > 0$. Then f^* is also a non-degenerate Orlicz function. It is called the function complementary to f. We have $(f^*)^* = f$. For any $u \geqq 0$, $v \geqq 0$ the Young's inequality $uv \leqq f(u) + f^*(v)$ holds (cf. [3], p. 147).

A sequence $\{f_n\}_{n=1}^\infty$ of Orlicz functions is said to satisfy the uniform Δ_2-condition if there exists $K > 0$ and n_0 such that we have $f_n(2t) \leqq K f_n(t)$ for each $t \in \left\langle 0, \frac{1}{2}\right\rangle$ and $n \geqq n_0$ ([3], p. 167).

In what follows denote by s the linear space of all sequences of real numbers. Denote by d the metric on s defined in the following way:

$$d(x, y) = \sum_{k=1}^\infty \frac{1}{2^k} \frac{|\xi_k - \eta_k|}{1 + |\xi_k - \eta_k|} \tag{1}$$

$$(x = \{\xi_k\}_{k=1}^\infty \in s, \quad y = \{\eta_k\}_{k=1}^\infty \in s).$$

Further denote by l_∞ and c_0 the linear space of all bounded sequences of real numbers and all sequences of real numbers converging to 0, respectively, each with the norm

$$\|x\| = \sup_{k=1,2,\dots} |\xi_k| \qquad (x = \{\xi_k\}_{k=1}^\infty)$$

2. The modular sequence space determined by a sequence of Orlicz functions

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-degenerate Orlicz functions. For $x = \{\xi_k\}_{k=1}^{\infty} \in s$ we put $\varrho(x) = \sum_{n=1}^{\infty} f_n(|\xi_n|)$ and

$$l\{f_n\} = \left\{x \in s: \ \underset{t>0}{\exists} \ \varrho(tx) < +\infty\right\}$$

Then ϱ is a convex modular on s and $l\{f_n\}$ is a modular space which is a Banach space ([3], p. 166; [4]). If the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the uniform Δ_2-condition, then

(2) $$l\{f_n\} = \left\{x \in s: \ \underset{t>0}{\forall} \ \varrho(tx) < +\infty\right\}$$

and

(3) $$l\{f_n\} \subset c_0$$

(cf. [8]).

In general, $f_n(1) \neq 1$. If $f_n(a_n) = 1$ for $a_n > 0$, then we can put $g_n(t) = f_n(a_n t)$. So $g_n(1) = 1$, $\{g_n\}_{n=1}^{\infty}$ is a sequence of Orlicz functions. Moreover, $l\{f_n\}$ and $l\{g_n\}$ are isometric spaces ([8]).

Proposition 2.1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-degenerate Orlicz functions. If $f_n(1) = 1$ $(n = 1, 2, \ldots)$ and $\sum_{n=1}^{\infty} f_n(t) < +\infty$ for some $t > 0$, then $l\{f_n\} = l_{\infty}$.

Proof. Let $x = \{\xi_n\}_{n=1}^{\infty} \in l\{f_n\}$, $\varepsilon > 0$. Then from the inequality

$$\varrho\left(\frac{x}{\|x\| + \varepsilon}\right) \leqq 1$$

we get

$$f_n\left(\frac{|\xi_n|}{\|x\| + \varepsilon}\right) \leqq 1 \qquad (n = 1, 2, \ldots)$$

So we have $|\xi_n| \leqq \|x\|$ $(n = 1, 2, \ldots)$. Hence $x \in l_{\infty}$.

Let $x = \{\xi_n\}_{n=1}^{\infty} \in l_{\infty}$. Then there exists $K > 0$ such that $|\xi_n| \leqq K$ $(n = 1, 2, \ldots)$. According to the assumption there exists a $t_0 > 0$ such that $\sum_{n=1}^{\infty} f_n(t_0) < +\infty$. But then for a suitable $t_1 > 0$ we have

$$|t_1 \xi_n| \leqq t_1 K \leqq t_0 \qquad (n = 1, 2, \ldots)$$

and hence

$$\varrho(t_1 x) = \sum_{n=1}^{\infty} f_n(|t_1\xi_n|) \leqq \sum_{n=1}^{\infty} f_n(t_0) < +\infty,$$

therefore $x \in l\{f_n\}$.

In the following theorem the set s is considered as a metric space with the metric d defined in (1).

Theorem 2.1. Let the sequence $\{f_n\}_{n=1}^{\infty}$ of non-degenerate Orlicz functions satisfy the uniform Δ_2-condition. The set $l\{f_n\}$ is a dense F_σ-set of the first Baire category in s.

Proof. According to Theorem 2,1 from [1] the set $c_0 \subset s$ is a set of the first Baire category in s. Using the inclusion (3) we see that $l\{f_n\}$ is a set of the first category in s, too.

For $m, k \in N$ we put

$$A_{mk} = \left\{x = \{\xi_n\}_{n=1}^{\infty} \in s: \sum_{j=1}^{m} f_j(|\xi_j|) \leqq k\right\}$$

Then A_{mk} is a closed set in s and

$$l\{f_n\} = \bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} A_{mk}$$

Therefore $l\{f_n\}$ is an F_σ-set in s. The density of $l(\{f_n\}$ in s is obvious.

A sequence $\{h_n\}_{n=1}^{\infty}$ of Orlicz functions is said to satisfy the condition (P) if for each $t > 0$ we have $\sum_{n=1}^{\infty} h_n(t) = +\infty$, further $h_n(1) = 1$ $(n = 1, 2, \ldots)$ and $\{h_n\}_{n=1}^{\infty}$ satisfies the uniform Δ_2-condition.

Theorem 2.2. Let the sequences $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ of non-degenerate Orlicz functions satisfy the condition (P). If $l\{f_n\} \cap l\{g_n\} \neq l\{f_n\}$, then the set $l\{f_n\} \cap l\{g_n\}$ is a dense F_σ-set of the first Baire category in $l\{f_n\}$.

Proof. Denote by ϱ and $\|\ \|$ the modular and the norm introduced by the sequence $\{f_n\}_{n=1}^{\infty}$ of Orlicz functions.

Each sequence with only a finite number of non-zero terms belongs to $l\{f_n\} \cap l\{g_n\}$. The set of all such sequences is dense in $l\{f_n\}$. Therefore $lf_n\} \cap l\{g_n\}$ is dense in $l\{f_n\}$.

We shall prove that $l\{f_n\} \cap l\{g_n\}$ is an F_σ-set of the first category in $l\{f_n\}$. Put

$$C_k = \bigcup_{n=1}^{\infty} \left\{x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}: \sum_{j=1}^{n} g_j(|\xi_j|) > k\right\} \qquad (k = 1, 2, \ldots).$$

Since $l\{f_n\} \cap l\{g_n\} \neq l\{f_n\}$, there exists such an $x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}$ that $x \notin l\{g_n\}$.

Hence (cf. [8]) we have $\sum_{j=1}^{\infty} g_j(|\xi_j|) = +\infty$ and so

(4) $$\bigcap_{k=1}^{\infty} C_k \neq \emptyset$$

Let $x_0 = \{\xi_j^0\}_{j=1}^{\infty} \in C_k$. Then there exists an n such that

$$\sum_{j=1}^{n} g_j(|\xi_j^0|) > k$$

Choose $\varepsilon > 0$ such that

$$\sum_{j=1}^{n} g_j(|\xi_j^0|) - \varepsilon > k$$

Since the functions $g_1, g_2, \ldots, g_n$ are continuous, there is a $\delta_1 > 0$ such that

(5) $$|\xi_j - \xi_j^0| < \delta_1 \quad (j = 1, \ldots, n) \Rightarrow \sum_{j=1}^{n} |g_j(|\xi_j^0|) - g_j(|\xi_j|)| < \varepsilon$$

Put $\delta = \min_{1 \leqq j \leqq n} f_j(\delta_1)$ and let $x = \{\xi_j\}_{j=1}^{\infty}$ be such that $\varrho(x - x_0) < \delta$, i.e. $\sum_{j=1}^{\infty} f_j(|\xi_j - \xi_j^0|) < \delta$. Then for each $j = 1, \ldots, n$ we have $f_j(|\xi_j - \xi_j^0|) < f_j(\delta_1)$ and so we get $|\xi_j - \xi_j^0| < \delta_1$ $(j = 1, 2, \ldots, n)$. Then according to (5) we have

$$\sum_{j=1}^{n} |g_j(|\xi_j|) - g_j(|\xi_j^0|)| < \varepsilon$$

and

$$\sum_{j=1}^{n} g_j(|\xi_j|) \geqq \sum_{j=1}^{n} g_j(|\xi_j^0|) - \sum_{j=1}^{n} |g_j(|\xi_j^0|) - g_j(|\xi_j|)| > \sum_{j=1}^{n} g_j(|\xi_j^0|) - \varepsilon > k$$

Hence $x \in C_k$. Thus we have proved that

$$\{x \in l\{f_n\}: \varrho(x - x_0) < \delta\} \subset C_k$$

According to Lemma 1.1 the set C_k is open.

We shall show that the set $C = \bigcap_{k=1}^{\infty} C_k$ is a dense set in $l\{f_n\}$.

Let $x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}$ and $0 < \varepsilon < 1$. We shall show that there is a $z \in C$ such that

(6) $$\|x - z\| < \varepsilon$$

Choose a $y = \{\eta_j\}_{j=1}^{\infty} \in C$ (see (4)). Then there is an n such that

$$\sum_{j=n+1}^{\infty} f_j\left(\frac{2|\xi_j|}{\varepsilon}\right) < 1, \quad \sum_{j=n+1}^{\infty} f_j\left(\frac{2|\eta_j|}{\varepsilon}\right) < 1$$

Put $z = \xi_1, \ldots, \xi_n, \eta_{n+1}, \eta_{n+2}, \ldots, \eta_{n+k}, \ldots$ It is easy to see that $z \in C$. Further we have

$$\varrho\left(\frac{x-z}{\varepsilon}\right) = \sum_{j=n+1}^{\infty} f_j\left(\frac{|\xi_j - \eta_j|}{\varepsilon}\right) \leqq \frac{1}{2} \sum_{j=n+1}^{\infty} f_j\left(\frac{2|\xi_j|}{\varepsilon}\right) + \frac{1}{2} \sum_{j=n+1}^{\infty} f_j\left(\frac{2|\eta_j|}{\varepsilon}\right) < 1$$

From this we get (6).

Thus C is a G_δ-set dense in $l\{f_n\}$. Therefore C is a residual set in $l\{f_n\}$ (cf. [7], p. 49). Hence

$$l\{f_n\} \cap l\{g_n\} = l\{f_n\} \setminus \bigcap_{k=1}^{\infty} C_k$$

is an F_σ-set of the first Baire category in $l\{f_n\}$. This ends the proof.

If $f_n = f$ $(n = 1, 2, \ldots)$, then $l\{f_n\}$ is an Orlicz sequence space and we denote it by l_f. In particular, if $f_n(t) = t^p$ $(p \geqq 1)$ for $n = 1, 2, \ldots,$ we have $l\{f_n\} = l^p$. Moreover, in this case the norm given by the modular on $l\{f_n\}$ coincides with the classical norm on l^p. Therefore from Theorem 2.2 the following results follow:

Corollary.

a) Let f and g be Orlicz functions satisfying the Δ_2-condition for small t and $l_f \cap l_g \neq l_f$. Then $l_f \cap l_g$ is a dense F_σ-set of the first Baire category in l_f.

b) If $1 \leqq p < q$, then l^p is a dense F_σ-set of the first Baire category in l^q (see [6]).

c) If a sequence $\{f_n\}_{n=1}^{\infty}$ of Orlicz functions satisfies the uniform Δ_2-condition, then it follows from [8] (Proposition 3.2) that $l\{f_n\} \subset l^p$ for some $p > 1$. Hence if $\{f_n\}_{n=1}^{\infty}$ satisfies the condition (P), there exists $p_0 > 1$ such that $l\{f_n\}$ is a dense F_σ-set of the first Baire category in l^p for each $p \geqq p_0$.

Theorem 2.3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-degenerate Orlicz functions. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers with $\limsup_{n\to\infty} a_n = +\infty$. Then the set

$$A = \{x = \{\xi_k\}_{k=1}^{\infty} \in l\{f_n\} \colon \limsup_{n\to\infty} a_n|\xi_n| < +\infty\}$$

is a dense F_σ-set of the first Baire category in $l\{f_n\}$.

Proof. Put

$$C_k = \bigcup_{n=1}^{\infty} \{x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\} \colon a_n|\xi_n| > k\} \qquad (k = 1, 2, \ldots).$$

We shall prove that C_k $(k = 1, 2, \ldots)$ is an open set in $l\{f_n\}$.

Let $x_0 = \{\xi_j^0\}_{j=1}^{\infty} \in C_k$. Then there exists an n such that $\alpha_n |\xi_n^0| > k$. Choose $\varepsilon > 0$ in such a way that $\alpha_n |\xi_n^0| - \varepsilon > k$. Put $\delta_1 = \frac{\varepsilon}{\alpha_n}$, $\delta = f_n(\delta_1)$. Let us suppose that $x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}$ satisfies the condition $\varrho(x - x_0) < \delta$. Then $\sum_{j=1}^{\infty} f_j(|\xi_j - \xi_j^0|) < \delta$. From this we get $f_n(|\xi_n - \xi_n^0|) < \delta = f_n(\delta_1)$ and therefore $|\xi_n - \xi_n^0| < \delta_1$. Hence $\alpha_n |\xi_n - \xi_n^0| < \varepsilon$ and so

$$\alpha_n |\xi_n| \geqq \alpha_n |\xi_n^0| - \alpha_n |\xi_n - \xi_n^0| > \alpha_n |\xi_n^0| - \varepsilon > k.$$

So we have proved that $\{x: \varrho(x - x_0) < \delta\} \subset C_k$. According to Lemma 1.1 the set C_k $(k = 1, 2, \ldots)$ is open in $l\{f_n\}$.

We shall show that the set $C = \bigcap_{k=1}^{\infty} C_k$ is a dense set in $l\{f_n\}$.

At first we shall show that $C \neq \emptyset$. Since $\limsup_{n \to \infty} \alpha_n = +\infty$ and $\lim_{t \to 0+} f_j(t) = 0$ for each $j = 1, 2, \ldots$, we can choose a sequence $n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$ of positive integers such that $\alpha_{n_k} > k^2$ $(k = 1, 2, \ldots)$ and

$$\sum_{k=1}^{\infty} f_k((\sqrt{\alpha_{n_k}})^{-1}) < +\infty$$

Put $\xi_{n_k}^0 = (\sqrt{\alpha_{n_k}})^{-1}$ $(k = 1, 2, \ldots)$ and $\xi_j^0 = 0$ for $j \neq n_k$ $(k = 1, 2, \ldots)$. Then $x_0 = \{\xi_j^0\}_{j=1}^{\infty} \in C$.

Let $x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}$, $\varepsilon > 0$. Since $x, x_0 \in l\{f_n\}$, there exists an m such that

$$\sum_{j=m+1}^{\infty} f_j\left(\frac{2|\xi_j|}{\varepsilon}\right) < 1, \quad \sum_{j=m+1}^{\infty} f_j\left(\frac{2|\xi_j^0|}{\varepsilon}\right) < 1 \tag{7}$$

Choose $y = \{\eta_j\}_{j=1}^{\infty}$ in the following way: $\eta_j = \xi_j$ for $j \leqq m$ and $\eta_j = \xi_j^0$ for $j > m$. Then $y \in C$ and on account of (7) we have

$$\sum_{j=1}^{\infty} f_j\left(\frac{|\xi_j - \eta_j|}{\varepsilon}\right) = \sum_{j=m+1}^{\infty} f_j\left(\frac{|\xi_j - \xi_j^0|}{\varepsilon}\right) \leqq \sum_{j=m+1}^{\infty} f_j\left(\frac{1}{2}\frac{2|\xi_j|}{\varepsilon} + \frac{1}{2}\frac{2|\xi_j^0|}{\varepsilon}\right) \leqq$$
$$\leqq \frac{1}{2}\sum_{j=m+1}^{\infty} f_j\left(\frac{2|\xi_j|}{\varepsilon}\right) + \frac{1}{2}\sum_{j=m+1}^{\infty} f_j\left(\frac{2|\xi_j^0|}{\varepsilon}\right) < 1$$

Hence $\|y - x\| \leqq \varepsilon$. The density of C in $l\{f_n\}$ follows.

The set C is a G_δ-set dense in $l\{f_n\}$, therefore it is a residual set in $l\{f_n\}$ ([7], p. 49).

It is easy to check that $A = l\{f_n\} \setminus C$. Hence A is an F_σ-set of the first Baire category in $l\{f_n\}$. The density of A in $l\{f_n\}$ is evident. This ends the proof.

Let $x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}$, $y = \{\eta_j\}_{j=1}^{\infty} \in l\{f_n^*\}$, f_n^* being the function complementary to f_n. Then it follows from the Young's inequality $\sum_{j=1}^{\infty} |\xi_j \eta_j| < +\infty$. For a sequence which does not belong to $l\{f_n^*\}$ we have the following result.

Theorem 2.4. Let $a = \{\alpha_j\}_{j=1}^{\infty} \notin l\{f_n^*\}$. Then the following assertions hold:

a) For all points $x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}$ excepting points of a certain F_σ-set of the first Baire category we have

$$\liminf_{m \to \infty} \sum_{j=1}^{m} \alpha_j \xi_j = -\infty, \quad \limsup_{m \to \infty} \sum_{j=1}^{m} \alpha_j \xi_j = +\infty \tag{8}$$

b) The set

$$M = \left\{ x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}: \sum_{j=1}^{\infty} |\alpha_j \xi_j| < +\infty \right\}$$

is an F_σ-set of the first Baire category in $l\{f_n\}$. Moreover, if $a \in l_\infty$, then M is a dense set.

Proof.

a) Put

$$C_k = \bigcup_{m=1}^{\infty} \left\{ x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}: \sum_{j=1}^{m} \alpha_j \xi_j > k \right\} \qquad (k = 1, 2, \ldots), \quad C = \bigcap_{k=1}^{\infty} C_k.$$

Every continuous linear functional φ on $l\{f_n\}$ is of the form $\varphi(x) = \sum_{j=1}^{\infty} \xi_j \eta_j$, where $y = \{\eta_j\}_{j=1}^{\infty} \in l\{f_n^*\}$ (cf. [8]). Since $a \notin l\{f_n^*\}$, there exists a point $x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}$ such that $\sum_{j=1}^{\infty} \alpha_j \xi_j = +\infty$. Therefore the set C is non-empty.

We shall prove that C_k $(k = 1, 2, \ldots)$ is an open set in $l\{f_n\}$. For $x = \{\xi_j\}_{j=1}^{\infty} \in C_k$ there exists an integer m such that $\sum_{j=1}^{m} \alpha_j \xi_j > k$. Choose numbers $\varepsilon > 0$, $\delta > 0$ such that

$$\sum_{j=1}^{m} \alpha_j \xi_j - \varepsilon > k, \quad \delta \sum_{j=1}^{m} |\alpha_j| < \delta$$

Put $\delta_1 = \min_{1 \leq j \leq m} f_j(\delta)$. For each $y = \{\eta_j\}_{j=1}^{\infty}$ satisfying $\varrho(y - x) < \delta_1$ we have $|\xi_j - \eta_j| < \delta$ $(j = 1, 2, \ldots, m)$. Therefore,

$$\sum_{j=1}^{m} \alpha_j \eta_j = \sum_{j=1}^{m} \alpha_j \xi_j + \sum_{j=1}^{m} \alpha_j (\eta_j - \xi_j) > \sum_{j=1}^{m} \alpha_j \xi_j - \varepsilon > k$$

Hence $\{y: \varrho(y - x) < \delta_1\} \subset C_k$ and according to Lemma 1.1 the set C_k is open in $l\{f_n\}$.

We shall show that the set $C = \bigcap_{k=1}^{\infty} C_k$ is dense in $l\{f_n\}$.

Let $x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}$ and $\varepsilon > 0$. Choose a fixed $y = \{\eta_j\}_{j=1}^{\infty} \in C$. Then there exists p such that

$$(9) \qquad \sum_{j=p+1}^{\infty} f_j\left(\frac{2|\xi_j|}{\varepsilon}\right) < 1, \quad \sum_{j=p+1}^{\infty} f_j\left(\frac{2|\eta_j|}{\varepsilon}\right) < 1$$

Put $t = \{\tau_j\}_{j=1}^{\infty}$, where $\tau_j = \xi_j$ for $j \leqq p$ and $\tau_j = \eta_j$ for $j > p$. Then $t \in C$ and using (9) we get

$$\varrho\left(\frac{x-t}{\varepsilon}\right) = \sum_{j=1}^{\infty} f_j\left(\frac{|\xi_j - \tau_j|}{\varepsilon}\right) \leqq \sum_{j=p+1}^{\infty} f_j\left(\frac{|\xi_j| + |\eta_j|}{\varepsilon}\right) \leqq \frac{1}{2} \sum_{j=p+1}^{\infty} f_j\left(\frac{2|\xi_j|}{\varepsilon}\right) +$$

$$+ \frac{1}{2} \sum_{j=p+1}^{\infty} f_j\left(\frac{2|\eta_j|}{\varepsilon}\right) < 1.$$

Hence $\|t - x\| \leqq \varepsilon$ and so C is a dense G_δ-set in $l\{f_n\}$. Therefore C is a residual set in $l\{f_n\}$.

Analogously we can show that also the set $D = \bigcap_{k=1}^{\infty} D_k$ is residual in $l\{f_n\}$, where

$$D_k = \bigcup_{m=1}^{\infty} \left\{x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}: \sum_{j=1}^{m} \alpha_j \xi_j < -k\right\} \qquad (k = 1, 2, \dots,).$$

Using the foregoing results we see that the set $C \cap D$ is a residual set in $l\{f_n\}$. It is obvious that this set coincides with the set of all $x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}$ for which (8) holds.

b) Let

$$H_k = \bigcup_{m=1}^{\infty} \left\{x = \{\xi_j\}_{j=1}^{\infty} \in l\{f_n\}: \sum_{j=1}^{m} |\alpha_j \xi_j| > k\right\} \qquad (k = 1, 2, \dots).$$

Using analogous arguments as in the part a) of the proof we can show that H_k $(k = 1, 2, \dots)$ are open sets in $l\{f_n\}$ and $H = \bigcap_{k=1}^{\infty} H_k$ is a dense set in $l\{f_n\}$. Since $M = l\{f_n\} \setminus H$, we see that M is an F_σ-set of the first category in $l\{f_n\}$.

If $a \in l_\infty$, then $l^1 \subset M$. According to Theorem 2.2 the set l^1 is dense in $l\{f_n\}$, which completes the proof.

Corollary. Let $a = \{\alpha_j\}_{j=1}^{\infty} \notin l^q$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the following assertion hold:

a) The set

$$\left\{x = \{\xi_j\}_{j=1}^{\infty} \in l^p: \sum_{j=1}^{\infty} |\alpha_j \xi_j| < +\infty\right\}$$

is a dense F_σ-set in l^p.

b) For all points $x = \{\xi_j\}_{j=1}^{\infty} \in l^p$ excepting the points of a certain F_σ-set of the first Baire category we have

$$\liminf_{m \to \infty} \sum_{j=1}^{m} a_j \xi_j = -\infty, \quad \limsup_{m \to \infty} \sum_{j=1}^{m} a_j \xi_j = +\infty$$

([3], Theorem 3.1).

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SÚHRN

APLIKÁCIE METÓDY KATEGORIÍ V TEÓRII MODULÁRNYCH PRIESTOROV POSTUPNOSTÍ

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V práci sa študuje štruktúra modulárnych priestorov postupností z hľadiska Baireových kategórií množín. Niektoré výsledky práce zovšeobecňujú skoršie výsledky druhého z autorov.

РЕЗЮМЕ

ПРИМЕНЕНИЯ МЕТОДА КАТЕГОРИЙ В ТЕОРИИ МОДУЛЯРНЫЦХ ПРОСТРАНСТВ ПОСЛЕДОВАТЕЛЬНОСТЕЙ

Янина Эверт, Слупск — Тибор Шалат, Братислава

В работе исследована структура модулярных пространств последовательностей с точки зрения беровских категорий множеств. Некоторые результаты работы обобщают нредыдущие результаты второго из авторов.