UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XLVIII—XLIX — 1986

ON THE STRUCTURE OF CERTAIN FUNCTIONAL SPACES WITH BAIRE METRIC

ALBERT MARENČIN, Bratislava

The space of all real-value functions defined on the interval $[1, \infty)$, equipped with the Baire metric ω, where $\omega(f, g) = 0$ if $f = g$ and $\omega(f, g) = 1/\inf\{x, f(x) \neq g(x)\}$ if $f \neq g$, see [3, p. 67]. There are several papers dealing with special spaces of discrete functions with the Baire metric ([2], [4]), which have given an impetus for investigating the continuous case. In this paper, some structural questions on spaces of continuous functions defined on $[1, \infty)$ with the above metric are studied from the viewpoint of Category Theory and it is shown that in such a space there exists a residual set of functions having a derivative at least in one point, which, in a sense, is a contrary result as in the space with usual metric.

Structure of the space

Let Ω denote the space of continuous functions defined on the interval $[1, \infty)$. On this space, define the metric

$$\omega(f, g) = 0 \quad \text{if} \quad f = g$$
$$\omega(f, g) = 1/\inf\{x, f(x) \neq g(x)\} \quad \text{if} \quad f \neq g$$

for all $f, g \in \Omega$. Given any $\varepsilon > 0$, we can construct the open sphere $K(f, \varepsilon)$. Then for $g \in K(f, \varepsilon)$, $f \neq g$, we have $\omega(f, g) = 1/\inf\{x, f(x) \neq g(x)\} < \varepsilon$, that is, $\inf\{x, f(x) \neq g(x)\} > 1/\varepsilon$. It follows if there is $\delta > 0$ that $f(x) = g(x)$ for every $x \in [1, 1/\varepsilon + \delta)$, then $g \in K(f, \varepsilon)$.

Theorem 1. The metric space (Ω, ω) is complete.

Proof. Let a Cauchy sequence $\{f_n\}_1^\infty$ of functions be given in Ω. Then for every $\varepsilon > 0$ there is the least such index $m(\varepsilon)$ that for distinct $m, n \geqq m(\varepsilon)$ we have

$$\omega(f_m, f_n) > \varepsilon.$$

Obviously, if $f_m \neq f_n$, then $f_m(x) = f_n(x)$ for every $x \in [1, 1/\varepsilon]$. Put $\varepsilon = 1/x$ and define f by

$$f(x) = f_{m(\varepsilon)}(x).$$

It follows that $f \in \Omega$. Since for $m \geqq m(\varepsilon)$ we have

$$f_m(t) = f(t), \qquad t \in [1, 1/\varepsilon],$$

f is a limit function of the sequence $\{f_n\}_1^\infty$ in the metric ω, and thus Ω is a complete space.

Denote by U the set of all functions $h \in \Omega$ with

$$h(x) = \int_1^x f(t)\, \mathrm{d}t, \qquad f \in \Omega. \tag{1}$$

Theorem 2. The set U is closed in Ω.

Proof. Let $\{h_n\}_1^\infty$ be a sequence of functions in U. Assume that $h_n \to h$ as $n \to \infty$. We show that $h \in U$. Evidently, $\{h_n\}_1^\infty$ is a Cauchy sequence and each of its elements can be expressed in the form

$$h_n(x) = \int_1^x f_n(t)\, \mathrm{d}t, \qquad f_n \in \Omega \quad (n = 1, 2, \ldots).$$

To prove the theorem, it is sufficient to check that

$$\omega(h_m, h_n) = \omega(f_m, f_n) \tag{2}$$

$$\int_1^x f_m(t)\, \mathrm{d}t = \int_1^x f_n(t)\, \mathrm{d}t \tag{3}$$

if and only if $f_m(z) = f_n(z)$ for every $z \in [1, x_0]$. The "if" part of this assertion is trivial, the "only if" part follows easily from continuity of f_m and f_n. Now it is immediately verified that $\{f_n\}_1^\infty$ is a Cauchy sequence in Ω, and owing to completeness of Ω, it converges to some $f \in \Omega$. Putting

$$g(x) = \int_1^x f(t)\, \mathrm{d}t,$$

we have $g \in U$, and since $\omega(h_n, g) = \omega(f_n, f)$, we obtain $h_n \to g$. The uniqueness of a limit implies $g(x) = h(x)$. The proof is complete.

Theorem 3. The set U is nowhere dense in Ω.

Proof. It is sufficient to show that $\Omega - U$ is dense in Ω. Let $\varepsilon > 0$ and consider and open sphere $K(f, \varepsilon) \subset \Omega, f \in \Omega$. If suffices to find a function $g \in \Omega$ with $g \in \Omega - U$ and $\omega(f, g) > \varepsilon$. We may put

$$g(x) = f(x) \quad \text{for} \quad x \leqq 2/\varepsilon$$

$$g(x) = f(2/\varepsilon) + (x - 2/\varepsilon) \cos \frac{1}{x - 2/\varepsilon} \quad \text{for} \quad x > 2/\varepsilon.$$

It is evident that $g \in \Omega$ and it can be proved that g has not a finite variation on the interval $[2/\varepsilon, 2/\varepsilon + 1)$. Since every function $h \in U$ is absolutely continuous, it necessarily has a finite variation. Hence we infer that $g \notin U$ and $\inf \{x, f(x) \neq \neq g(x)\} \geqq 2/\varepsilon$, which implies $\omega(f, g) \leqq \varepsilon/2 < \varepsilon$. Now choose δ with $3/2\varepsilon < < 1/\delta < 2/\varepsilon$.

Thus $2\varepsilon/3 > \delta > \varepsilon/2$. Construct the open sphere $K(g, \delta) \subset \Omega$. The above reasoning yields that $U \cap (K(g, \delta) \cap \Omega) = \emptyset$. Therefore, $\Omega - U$ is dense in Ω. Since by Theorem 2, U is closed in Ω, we obtain that U is nowhere dense in Ω.

In the following theorem we prove that U includes a residual subset L of functions which are unbounded.

Theorem 4. The set $L = \{f \in U, \sup |f(x)| = +\infty, 1 \leqq x < \infty\}$ is residual in U.

Proof. Denote $A = U - L$ and put

$$A_n = \{f \in U, \sup |f(x)| < n, 1 \leqq x < \infty\}.$$

First we show that $A = \bigcup_{n=1}^{\infty} A_n$. Obviously, $\bigcup_{n=1}^{\infty} A_n \subset A$.

Let $f \in A$, then for some $\alpha > 0$ we have

$$\sup |f(x)| \leqq \alpha, \qquad 1 \leqq x < \infty.$$

Choose $n > \alpha$, then

$$\sup |f(x)| \leqq \alpha < n, \qquad 1 \leqq x < \infty,$$

hence $f \in A_n$. Now we prove that A_n $(n = 1, 2, \ldots)$ is closed in U. Consider a sequence of functions $\{f_k\}_1^\infty$ in A_n. Let $f_k \to f$ as $k \to \infty$. It follows from the convergence in the metric ω that for every $x \in [1, \infty)$ there exists the least index $k(x)$ with $f_{k(x)}(t) = f(t)$ for each $t \in [1, x]$ and $f_{k(x)} \in A_n$. We infer that A_n is closed in U $(n = 1, 2, \ldots)$. We are going to prove now that A is a first-category set. Choose $\varepsilon > 0$ and construct, for $f \in U$, the open sphere $K(f, \varepsilon) \subset U$. Define the function g by

$$g(x) = f(x) \quad \text{if} \quad x < 2/\varepsilon$$
$$g(x) = f(2/\varepsilon) + x - 2/\varepsilon \quad \text{if} \quad x \geqq 2/\varepsilon.$$

We see that $g \in U$, $\omega(f, g) \leqq \varepsilon/2 < \varepsilon$ and $\sup |g(x)| = +\infty$, $1 \leqq x < \infty$. Choose a δ with $1/\delta > 2/\varepsilon + f(1/\delta) + n$. Clearly, $\delta < \varepsilon/2$. Now we see that $g \notin A_n$. Then $A_n \cap (K(g, \delta) \cap U) = \emptyset$. Thus we have shown that the closed set A_n $(n = 1, 2, \ldots)$

is nowhere dense in U. Due to $A = \bigcup_{n=1}^{\infty} A_n$, the set A is of the first category in U, and since U is of the second category in itself, $U - A = L$ is residual in U.

Continuous functions without a derivative

Banach and Mazurkiewicz have proved that, in the space $C(a, b)$ with the usual metric, the set of functions having a derivative at least in one point is of the first category. In other words, the set of functions $f \in C(a, b)$ which have a derivative at no point of the interval (a, b) is a residual set of the second category in $C(a, b)$. The question arises whether this is the case also in the space equipped with the Baire metric. The following theorem gives a negative answer. Denote by N the set of functions $f \in \Omega$ having a derivative at no point of their domain.

Theorem 5. The set N is nowhere dense in Ω.

Proof. First we prove that N is closed in Ω. Let $\{f_n\}_1^{\infty}$ be a sequence of functions in N. Let $f_n \to f$ as $n \to \infty$. It follows from the convergence that for every $x \in [1, \infty)$ there is the least index $n(x)$ with

$$f_{n(x)}(t) = f(t), \qquad t \in [1, x].$$

Since $f_{n(x)} \in N$ for any $n(x)$, we get $f \in N$ and so N is closed in Ω. Now it is sufficient to prove that $\Omega - N$ is dense in Ω. Choose $\varepsilon > 0$ and construct, for any $f \in \Omega$, the open sphere $K(f, \varepsilon)$. Now construct $g \in \Omega$ with $\omega(f, g) < \varepsilon$ and $g \in N$ by

$$g(x) = f(x) \quad \text{if} \quad x < 2/\varepsilon$$
$$g(x) = f(2/\varepsilon) + \varepsilon x/2 - 1 \quad \text{if} \quad x \geqq 2/\varepsilon.$$

It is evident that $g \in \Omega$ and $\omega(f, g) \leqq \varepsilon$. Choosing δ with $2/\varepsilon < 1/\delta < 3/\varepsilon$, we have $\varepsilon\, 3 < \delta < \varepsilon\, 2$ and we may construct the open sphere $K(g, \delta) < \Omega$. Since $g'(x)$ exists for every $x > 2/\varepsilon$, we get $N \cap (K(g, \delta) \cap \Omega) = \emptyset$. Hence by N being closed in Ω we obtain the assertion of the theorem.

Corollary. The set of all functions having a derivative at in least one point of the interval $(1, \infty)$ is residual in Ω.

Structure of the space Σ

So far, we have considered only continuous functions on the interval $[1, \infty)$. Now let us study the functions bounded on $C_f \cap [1, x]$, $x \in [1, \infty)$ for which the set of discontinuity points is of Lebesgue measure 0. Denote this set of functions by Σ. Further, let C_f and D_f stand for the sets of continuity points or discontinuity points, respectively, of a function $f \in \Sigma$. A pseudometric σ can be introduced on Σ as follows. Let $f, g \in \Sigma$, then

$$\sigma(f, g) = 0 \quad \text{if} \quad f = g \quad \text{almost everywhere}$$
$$\sigma(f, g) = 1/\inf \{x \in C_f \cap C_g, f(x) \neq g(x)\} \quad \text{if} \quad f \neq g$$
$$\text{on a set of positive measure.}$$

The following theorem generalizes some results already proved for the space Ω (see [2]).

Theorem 6.

a) The pseudometric space (Σ, σ) is complete.

b) The set $V = \left\{h \in \Sigma, h(x) = \int_1^x f(t) \, dt, f \in \Sigma\right\}$ is closed and nowhere dense in Σ.

c) The set $K = \{f \in V, \sup |f(x)| = +\infty, x \in C_f\}$ is residual in V.

REFERENCES

[1] Čech, E.: Bodové množiny, Praha 1974.
[2] Kostyrko, P.: O nekotorych prostranstvach s metrikoj tipa Bera, Mat.-Fyz. časop. 1966, 143—153.
[3] Sikorski, R.: Funkcje rzeczywiste I, Warszawa 1958.
[4] Šalát, T.: O istých priestoroch s Bairovskou metrikou, Mat.-Fyz. časop. 1957, 193—206.

Author's address:
Albert Marenčin,
Jelenia 6
Bratislava
811 05

Recieved: 17. 10. 1983

SÚHRN

O ŠTRUKTÚRE NIEKTORÝCH FUNKCIONÁLNYCH PRIESTOROV

Albert Marenčin, Bratislava

Z hľadiska teórie kategórií sa vyšetruje štruktúra niektorých množín spojitých funkcií v priestore Ω všetkých spojitých reálnych funkcií definovaných na intervale $\langle 1, \infty)$ s Baireovou metrikou ϱ. Niektoré výsledky platia aj vo všeobecnejšom priestore Σ skoro všade spojitých, ohraničených reálnych funkcií definovaných na intervale $\langle 1, \infty)$ s presudometrikou σ. Ďalej je v tomto článku dokázaná veta, že v priestore Ω je riedka množina funkcií, ktoré nemajú deriváciu v žiadnom bode intervalu $\langle 1, \infty)$.

РЕЗЮМЕ

О СТРУКТУРЕ НЕКОТОРЫХ ФУНКЦИОНАЛЬНЫХ ПРОСТРАНСТВ

Алберт Маренчин, Братислава

С точки зрения теории категории исследуется структура некоторых множеств непрерывных функций в пространстве Ω, всех непрерывных, вещественных функций, определенных на интервале $\langle 1, \infty)$ с метрикой ϱ типа Бэра. Некоторые результаты имеют силу в пространстве Σ всех почти всюда непрерывных, ограниченных вещественных функций, определенных на интервале $\langle 1, \infty)$ с псэудометрикой σ. В дальшем доказано, что множество функций не имеющих деривацию в никакой точке интервала $\langle 1, \infty)$ никде не плотно в пространстве Ω.