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ON A UNIFYING PRINCIPLE IN REAL ANALYSIS

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In the paper [1] an interesting theorem, implying several applications in mathematical analysis of functions of one real variable was proved. The well-known theorems for continuous functions on a compact interval were obtained as corollaries. The ordering on the real line was the main tool in the proof of the mentioned theorem. The present paper follows the idea of [1] and gives some additional remarks. In fact we present two notes. The first one shows that the result of [1], (Theorem 1 below) concerning the ordering and subordering may be formulated in such a way, that it is independent of the connectedness of the ordering. The second difference from [1], if there is any, is a formulation which perhpas enables more direct applications at least in $\boldsymbol{R}^n$.

For the sake of completness, let us recall the idea contained in [1]. If $<$ denotes the natural ordering on an interval $I \subset (-\infty, \infty)$, then a relation $\mathscr{R} \subset I \times I$ is called a subordering of $<$, if it is transitive and contained in $<$. We write $x\mathscr{R}y$, $x\not\mathscr{R}y$, if $(x, y) \in \mathscr{R}$, $(x, y) \notin \mathscr{R}$ respectively. A subordering $\mathscr{R}$ is called locally valid (see [1]) at $c \in I$, if there is a neighborhood $V(c)$ such that for any $x \in V(c) - \{c\}$, $x < c$, we have $x\mathscr{R}c$ and for any $x \in V(c) - \{c\}$, $c < x$, we have $c\mathscr{R}x$. It is called locally valid, if it is locally valid at any $c \in I$.

Note that the notion of a locally valid subordering may be defined in a natural way also for the case of an ordering which is defined on a general topological space X. The next theorem was proved in [1]. It concerns the suborderings of the natural ordering $<$ on the real line.

Theorem 1. The natural ordering is the only subordering which is locally valid on the interval I.

In what follows X is a linear normed space and M a set, $M \subset X$. The symbol $<$ denotes any relation on M which is transitive and has the following property

$$\text{(P)} \qquad \text{If} \quad x, y \in M,\ x < y, \quad \text{then} \quad \frac{x+y}{2} \in M,\ x < \frac{x+y}{2} < y$$

Example 1. Let $X = \boldsymbol{R}^n$ and let M be an n-dimensional interval in $\boldsymbol{R}^n$. Put $x < y$ for $x, y \in \boldsymbol{R}^n$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ iff $x_i < y_i$ for $i = 1, 2, \ldots, n$. (We use the notation $<$ also for the natural ordering in $\boldsymbol{R}$.)

Example 2. We can take in Example 1 instead of the n-dimensional interval such an n-dimensional interval which contains only the points with rational coordinates.

Now let us consider some suborderings.

Example 3. (Cf. [1] Example 2). Let $<$ and M have the same meaning as in Example 1. Let $\{\Omega_a\}$ be an open covering of M. If $x, y \in M$ define $x\mathcal{R}y$ iff $x < y$ and the interval $[x, y]$ has a finite subcovering. Here $[x, y]$ is the set of all $(t_1, \ldots, t_n)$ for which $x_i \leqq t_i \leqq y_i$ $(i = 1, 2, \ldots, n)$.

Example 4. Let $<$, M have the same meaning as in Example 2. For the sake of simplicity take $n = 1$. Let $f\colon M \to \boldsymbol{R}$ be a real function with the following property. If $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ are nondecreasing and nonincreasing sequences respectively, such that $x_n < y_n$ $(n = 1, 2, \ldots)$ and such that $\lim_{n\to\infty} (x_n - y_n) = 0$, then $\lim_{n\to\infty} (f(x_n) - f(y_n)) = 0$. Define $\mathcal{R}$ as follows. $x\mathcal{R}y$ iff $x < y$ and simultaneously f is bounded on $[x, y]$.

Again let X be a normed space, $M \subset X$ and $<$ a transitive relation on M. A subordering $\mathcal{R} \subset <$ will be said to be sequentially valid (with respect to $<$) if for any two sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $x_n < y_n$, $x_n, y_n \in M$ $(n = 1, 2, \ldots)$ such that $\{x_n\}_{n=1}^{\infty}$ is nondecreasing, $\{y_n\}_{n=1}^{\infty}$ is nonincreasing and $\lim_{n\to\infty} \|x_n - y_n\| = 0$, there exists n_0 such that $x_{n0}\mathcal{R}y_{n0}$.

Example 5. As an example of a sequentially valid subordering we can take the suborderings in Example 3 and Example 4.

Theorem 2. Let X be a linear normed space. Let $M \subset X$ and $<$ be a transitive ordering on M satisfying (P). Then any sequentially valid subordering $\mathcal{R}$ of $<$ coincides with $<$.

Proof. Let $\mathcal{R}$ be any sequentially valid subordering of $<$. Suppose that there exist $x, y \in M$, $x < y$ with $x\not\mathcal{R}y$. Then putting $z = \frac{x + y}{2}$ we have $x < z < y$. Hence either $x\not\mathcal{R}z$ or $z\not\mathcal{R}y$. Keeping this in mind we may construct by induction sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ of elements of M such that the first is nondecreasing, the second nonincreasing, $x_n < y_n$ $(n = 1, 2, \ldots)$ $x_n\not\mathcal{R}y_n$ and $\|x_n - y_n\| = \frac{\|x - y\|}{2^n}$. Thus $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and we have $x_n\not\mathcal{R}y_n$ for $n = 1, 2, \ldots$ It is a contradiction.

Considering Example 3 one obtains as an application of Theorem 2 the Heine Borel Theorem for compact interval.

Using Example 4 one obtains that certain special continuous functions on the closed interval containing only rational points are bounded. One has only to apply Theorem 2. This result has nothing to do with Theorem 1. Note that the functions considered in Example 4 are the uniformly continuous ones. (See [4] for the proof.)

Of course it is not the aim of the paper to repeat the interesting examples similar or identical to those in [1]. Note that Theorem 2 may be applied to obtain various theorems for functions of several variables in a similar way as Theorem 1 was used for the functions of one variable. Some other applications may be perhaps obtained as indicated in Example 4.

The natural method of introducing the ordering in $\boldsymbol{R}^n$ such as in Example 1 leads to Proposition 2 which may be usefull. An easy proof is omitted. The symbols X_i, $(i = 1, 2, \ldots, n)$ denote normed spaces, $M_i \subset X_i$ subsets, $<_i$ relations on M_i satisfying (P) and $\mathcal{R}_i \subset <_i$ subordering. The space $X = X_1 \times \ldots \times X_n$ will be considered as a normed space in the usual way (see e.g. [2]).

Proposition 2. Let X_i, $M_i \subset X_i$, $\mathcal{R}_i$ $(i = 1, 2, \ldots, n)$ have the above meaning. Define $<$ and $\mathcal{R}$ on $M_1 \times \ldots \times M_n$ such that $x < y$ iff $x_i <_i y_i$ and $x \mathcal{R} y$ iff $x_i \mathcal{R}_i y_i$. Let $\mathcal{R}_i$ be sequentially valid (with respect to $<_i$), then $\mathcal{R}$ is sequentially valid (with respect to $<$).

Now we want to give some remarks to connections between sequentially valid and locally valid suborderings. Since the last was defined for the real line we restrict ourselves to the real line.

Proposition 3. If $<$ is the natural ordering on an interval $I \subset (-\infty, \infty)$, then a subordering $\mathcal{R} \subset <$ is locally valid if and only if it is sequentially valid.

Proof. It follows from Theorem 1 and Theorem 2 and from the fact that $<$ is both locally and sequentially valid.

In [1], Theorem 1 concerning the locally valid relation has been formulated also for linearly ordered topological spaces. It was proved that in this case any locally valid subordering coincides with the ordering if and only if the topological space is connected (see [2] pp. 57—58 for the definition).

We did not define something which is an analogy of sequentially valid subordering in a general linearly ordered topological space. Newertheless, one can see that Theorem 2 does not assume the connectedness. (Cf. Example 4.) Obviously, it is not a contradiction to the mentioned characterization of connected ordered spaces, because Theorem 2 assumes that (P) is satisfied.

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SÚHRN

O JEDNOM PRINCÍPE V REÁLNEJ ANALÝZE

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V práci sa podáva zo všeobecnejšieho hľadiska jeden spoločný princíp dôkazov niektorých základných viet analýzy.

РЕЗЮМЕ

ОБ ОДНОМ ПРИНЦИПЕ ВО ВЕЩЕСТВЕННОМ АНАЛИЗЕ

Анна Нойбрунова, Братислава

В работе обобщается один метод доказательств некоторых фундаментальных теорем анализа.