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TWO NOTES ON THE CONGRUENCE LATTICE OF THE p-ALGEBRAS

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I. Introduction

Answering a question of G. Grätzer [1] (problem 57), T. Katriňák in [3] gave a characterization of the congruence lattice of a modular p-algebra in terms of congruence pairs. One object of this paper is to continue this generalization characterizing the congruence lattice of the quasi-modular p-algebra. The variety of quasi-modular p-algebras properly contains the variety of modular p-algebras. It was shown in [6] that every quasimodular p-algebra L can be uniquely determined by the triple consisting of the Boolean algebra of closed elements of L, the lattice of dense elements of L and a suitable homomorphism.

Our second aim in this paper is to discuss the notion of permutability of congruences of the p-algebra satisfying the identity $x = x^{**} \wedge (x \vee x^*)$, and to show that the n-permutability of congruences of the p-algebra depends completely on the n-permutability of their restrictions on the lattice of dense elements.

II. Preliminaries

A *p-algebra* is an algebra $\langle L; \vee, \wedge, *, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ where $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded lattice such that for every $a \in L$ the element a^* is the pseudocomplement of a, i.e. $x \leq a^*$ iff $a \wedge x = 0$.

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The following rules of computation (see e. g. [1]) can be used frequently in any p-algebra

(1) $a \leq b$ implies $b^* \leq a^*$,

(2) $a \leq b^{**}$,

(3) $a^* = a^{***}$,

(4) $a^* \wedge a^{**} = 0$,

(5) $(a \vee b)^* = a^* \wedge b^*$.

The relation γ defined as $a \equiv b(\gamma)$ iff $a^* = b^*$ is the *Glivenko* congruence of L.

A p-algebra is said to be *quasi-modular* if $[(x \wedge y) \vee z^{**}] \wedge x = (x \wedge y) \vee (z^{**} \wedge x)$ for all $x, y, z \in L$. This variety properly contains the variety of modular p-algebras and is properly contained in the variety defined by $x = x^{**} \wedge (x \vee x^*)$. The standard results on quasi-modular p-algebras can be found in [4], [5], [6].

In any p-algebra L, define the set $B(L) = \{x \in L; x = x^{**}\}$ of *closed* elements. $\langle B(L); \vee, \wedge, {}^*, 0, 1\rangle$, where $a \vee b = (a^* \wedge b^*)^*$ forms a Boolean algebra. The set of *dense* elements $D(L) = \{x \in L; x^* = 0\}$ is a filter (dual ideal) in L. For an arbitrary lattice L the set $F(L)$ of all filters of L ordered under the set inclusion is a lattice. $x \vee x^* \in D(L)$ for every $x \in L$.

In the quasi-modular p-algebra L, consider the mapping $\varphi(L): B(L) \mapsto F(D(L))$ such that $a\varphi(L) = \{x \in D(L): x \geq a^*\} = [a^*) \wedge D(L)$, $a \in B(L)$. The mapping $\varphi(L)$ is a $\{0, 1, \vee\}$ — homomorphism (see [6, Theorem 3]). The triple $\langle B(L), D(L), \varphi(L)\rangle$ determines uniquely every quasi-modular p-algebra [6, Theorem 4].

Let L be a quasi-modular p-algebra and $\theta \in \mathrm{Con}(L)$, θ_B, θ_D denote the restriction of θ to $B(L)$, $D(L)$ respectively. Evidently $(\theta_B, \theta_0) \in \mathrm{Con}(B(L)) \times \mathrm{Con}(B(L))$. An arbitrary pair $(\theta_1, \theta_2) \in \mathrm{Con}(B(L)) \times \mathrm{Con}(D(L))$ will be called a *congruence pair* iff $a \in B(L)$, $u \in D(L)$, $u \geq a$ and $a \equiv 1(\theta_1)$ imply $u \equiv 1(\theta_2)$.

In [5, Theorem 1] the following result has been proved.

Theorem A.

Let L be a quasi-modular p-algebra. Then every congruence relation θ of L determines a congruence pair (θ_B, θ_D). Conversely, every congruence pair (θ_1, θ_2) uniquely determines a congruence relation θ on L with $\theta_B = \theta_1$ and $\theta_D = \theta_2$ such that the following conditions are equivalent:

(a) $x \equiv y(\theta)$ $(x, y \in L)$

(b) (i) $x^{**} \equiv y^{**}(\theta_1)$ and (ii) $x \vee x^* \equiv y \vee y^*(\theta_2)$;

(c) (i) $x^* \equiv y^*(\theta_1)$ and (ii) $x \vee d \equiv y \vee d(\theta_2)$ for all $d \in D(L)$.

III. A characterization of the congruence lattice of the quasimodular p-algebra

In the sequel Δ_L, ∇_L will denote the identical and the universal congruence of L respectively.

Theorem 1.

Let $\langle B; \wedge, \vee, ', 0, 1\rangle$ be a Boolean algebra. Let $\langle D: \vee, \wedge, 1\rangle$ be a lattice with unit, and let A be a subset of $\mathrm{Con}(B) \times \mathrm{Con}(D)$. Then A is the set of all congruence pairs of a quasi-modular p-algebra L with $B(L) = B$ and $D(L) = D$ if and only if the following conditions are fulfilled:

(i) A is a join complete sublattice of $\mathrm{Con}(B) \times \mathrm{Con}(D)$ containing (Δ_B, Δ_D);

(ii) For every $a \in B$ there exists a filter $F_a \in F(D)$ such that $(\theta[(a]], \Phi) \in A$ iff $\Phi \in [\theta[F_a], \nabla_D]$;

(iii) $F_a \wedge T_{a'}$ is a principal filter for every $a \in B$;

(iv) The map $a \mapsto F_a$ is a $\{0, 1, \vee\}$ — homomorphism of B into $F(D)$;

(v) For elements a, b, c, $g \in B$ and x, y, $z \in D$ let $a \wedge b \geqslant c$ and let $(F_a \vee [x)) \wedge (F_b \vee [y)) \supseteq F_c \vee [z]$ in $F(D)$. Then $(F_a \vee [x)) \wedge (F_b \vee [y)) \wedge (F_g \vee F_c \vee [z)) = ((F_a \vee [x)) \wedge (F_b \vee [y)) \wedge F_g) \vee F_c \vee [z)$.

Before proving Theorem 1 we shall estiblish the following:

Lemma 1.

Let A, B, D have the same meaning as in Theorem 1. Then the conditions (i) and (ii) of Theorem 1 imply:

(ii)′ For every $\psi \in \mathrm{Con}(B)$ there exists a filter $F_\psi \in F(D)$ such that $(\psi, \Phi) \in A$ iff $\Phi \in [\theta[F_\psi], \nabla_D]$. Moreover, if $\psi = \theta[J]$, J is an ideal of B, then $F_\psi = \vee(F_a: a \in J)$.

The proof is the same as that given in the first part of proving the sufficiency of the conditions of Theorem 1 in [3].

Proof of Theorem 1.

Since the proof of Theorem 1 is essentially the same as that of [3, Theorem 1], We shall omit the details.

Necessity. Let $A(L)$ be the set of all congruence pairs of a quasi-modular p-algebra L with $B(L) = B$ and $D(L) = D$. Let $\{(\psi_i, \Phi_i): i \in I\} \subseteq A(L)$ for $I \neq \emptyset$, conditions (i), (ii) and (iii) can be proved similarly as those of [3, Theorem 1]. For (iv), it is proved in [6, Theorem 3] that the map $\varphi(L): B \mapsto F(D)$ is a $\{0, 1, \vee\}$-homomorphism. For (v), it is proved in [6, 6.3] that every quasi-modular p-algebra satisfies condition (v).

Sufficiency. We need to show the existence of a quasimodular p-algebra L such that $B(L)=B$ and $D(L)=D$. Let B be non trivial. By (iii), (iv) and (v) $\langle B, D, \varphi\rangle$ form a quasi-modular triple in the sense of [6], and by [6, Theorem 4] there exists a quasi-modular p-algebra L such that $B(L)=B$, $D(L)=D$ and $\varphi(L)=\varphi$. Let $A(L)$ denote the lattice of all congruence pairs of L, then similar to [3, Theorem 1] we get $A(L)=A$.

Definition 1.

A quasi-modular p-algebra is said to be a quasi-modular *s-algebra* if it satisfies the identity $x^* \vee x^{**}=1$.

Corollary 1.

Let B be a Boolean algebra, D a lattice with 1, A a subset of $\mathrm{Con}\,(B)\times \mathrm{Con}\,(D)$. Then A is the set of all congruence pairs of a quasi-modular s-algebra L with $B(L)=B$, $D(L)=D$ if and only if (i), (ii), (iv) and (v) from Theorem 1 together with the condition:

(iii)′ $F_a \wedge F_{a'}=[1)$ for every $a \in B$;

are fulfilled.

Theorem 2.

The following systems of conditions are independent.

1) (i)—(v) from Theorem 1.

2) (i), (ii), (iii)′, (iv), (v) from Corollary 1.

Proof.

In order to prove the independence of (i)—(iv) and (iii)′ it is enough to take examples from the proof of [3, Theorem 2]. One only have to verify that the examples satisfy condition (v). But this is straightforward. Now, we shall construct two examples, the first satisfies (i)—(iv) but not (v), and the second satisfies (i), (ii), (iii)′, (iv) but not (v).

Example 1. Let B denote the eight-element Boolean algebra $\{0, a, b, c, a', b', c', 1\}$ and let D be pentagon $\{v, x, y, z, 1\}$, i.e. $v<x<y<1$, $v<z<1$. First we define a map $m \mapsto F_m$ from B onto $F(D)$ as follows:

$F_1=F_{b'}=F_{a'}=[v)$, $F_a=F_{c'}=[x)$, $F_b=[y)$, $F_c=[z)$ and $F_0=[1)$, where a, b, c are atoms of B. It is easily checked that $m \mapsto F_m$ is a $\{0, 1, \vee\}$-homomorphism from B onto $F(D)$. Since D is finite, $F_m \wedge F_{m'}$ is principal. Thus we have verified (iii) and (iv). Define now $A \subseteq \mathrm{Con}\,(B)\times \mathrm{Con}\,(D)$ as follows: $A=\{(\theta[(m]], \Phi)\in \mathrm{Con}\,(B)\times \mathrm{Con}\,(D)\colon \Phi \in [\theta[F_m], \nabla_D]\}$. Clearly, $(\Delta_B, \Delta_D)=(\theta[(0]], \Delta_D)\in A$. Moreover, $(\theta[(m]], \Phi_1)\vee(\theta[(n]], \Phi_2)=(\theta[(m\vee n]], \Phi_1\vee\Phi_2)$. But $\Phi_1\vee\Phi_2 \geqslant \theta[F_m]\vee\theta[F_n]=\theta[F_m\vee F_n]=\theta[F_{m\vee n}]$ for elements from A. In the same way one can show that A is closed with respect to the meet, because $F_{m\wedge n}\leqslant F_m\wedge F_n$. Hence A

satisfies also (i) and (ii). The condition (v) fails, because $F(D)$ is non-modular and $m \mapsto F_m$ maps B onto $F(D)$.

Example 2. Let B be the four-eslement Boolean algebra $\{0, a, a', 1\}$, and let D be the pentagon $\{v, x, y, z, 1\}$, where $x<y$. Define a map $m \mapsto F_m$ from B into $F(D)$ as follows: $F_a=[y)$, $F_{a'}=[z)$, $F_0=[1)$, $F_1=[v)$. This mapping satisfies conditions (iii)′ and (iv). Define

$$A=\{(\theta[(m]], \Phi) \in \mathrm{Con}\,(B) \times \mathrm{Con}\,(D) \colon \Phi \in [\theta[F_m], \nabla_D]\}\ .$$

It can be checked that A is a sublattice of $\mathrm{Con}\,(B) \times \mathrm{Con}\,(D)$ with $(\Delta_B, \Delta_D) \in A$. Thus (i) and (ii) are satisfied. Condition (v) is not satisfied, since if we take elements b, c, h, g from B such that $b=c=h=a$, $g=a'$ and elements t, p, f from D with $t=p=x$, $f=y$, then $b \wedge c \geqslant h$ and $(F_b \vee [t)) \wedge (F_c \vee [p)) \supseteq F_h \vee [f)$. Moreover, $(F_b \vee [t)) \wedge (F_c \vee [p)) \wedge (F_g \vee F_h \vee [f)) = [x)$ and $[(F_b \vee [t)) \wedge (F_c \vee [p)) \wedge F_g] \vee F_h \vee [f) = [y)$ but $[x) \neq [y)$ which means that the condition (v) is not satisfied.

IV. Permutability of congruences of p-algebras

The product $\theta \circ \Phi$ of two congruences θ, Φ of an algebra A is defined by the following rule: $a \equiv b(\theta \circ \Phi)$ iff there exists an element $c \in A$ such that $a \equiv c(\theta)$ and $c \equiv b(\Phi)$. Two congruences θ, Φ are said to be *permutable* iff $\theta \circ \Phi = \Phi \circ \theta$. Two congruences θ_1 and θ_2 are *n-permutable* iff $\theta_1 \circ \theta_2 \circ \theta_1 \circ \theta_2 \circ \ldots = \theta_2 \circ \theta_1 \circ \theta_2 \circ \theta_1 \circ \ldots$, where on both sides there are n-factors. An algebra is n-permutable if every two congruences in A are n-permutable. It is well known that an n-permutable algebra is $(n+1)$-permutable, (see [2], [7]).

Lemma 2.

Let θ and Φ be congruences on a p-algebra L, Then
(i) $(\theta \circ \Phi \circ \theta \circ \Phi \circ \ldots)_B = \theta_B \circ \Phi_B \circ \theta_B \circ \ldots$ (n times)
(ii) $(\theta \circ \Phi \circ \theta \circ \ldots)_D = \theta_D \circ \Phi_D \circ \theta_D \circ \ldots$ (n times)

Proof.

(i) Since θ_B, Φ_B are restrictions of θ, Φ with respect to $B(L)$, we see that $\theta_B \circ \Phi_B \circ \theta_B \circ \ldots \subseteq (\theta \circ \Phi \circ \theta \circ \ldots)_B$. Conversely, take a, $b \in B(L)$ such that $a \equiv b(\theta \circ \Phi \circ \theta \circ \ldots)$, i.e. $a \equiv b((\theta \circ \Phi \circ \theta \circ \ldots)_B)$, then there exist elements $c_1, \ldots, c_{n-1} \in L$ such that $a \equiv c_1(\theta)$, $c_1 \equiv c_2(\Phi)$, $\ldots$, $c_{n-1} \equiv b(\theta)$, if n is odd or $c_{n-1} \equiv b(\Phi)$ if n is even. Therefore $a = a^{**} \equiv c_1^{**}(\theta)$, $c_1^{**} \equiv c_2^{**}(\Phi)$, $\ldots$, $c_{n-1}^{**} \equiv b^{**} = b(\theta)$ or $c_{n-1}^{**} \equiv b^{**} = b(\Phi)$, for n odd or even. Hence $a \equiv b(\theta_B \circ \Phi_B \circ \theta_B \circ \ldots)$ and $(\theta \circ \Phi \circ \theta \circ \ldots)_B \subseteq (\theta_B \circ \Phi_B \circ \theta_B \circ \ldots)$.

(ii) Again as above we have only to prove $(\theta \circ \Phi \circ \theta \circ \ldots)_D \subseteq \theta_D \circ \Phi_D \circ \theta_D \circ \ldots$. Take $a, b \in D(L)$, assume $a \equiv b(\theta \circ \Phi \circ \theta \circ \ldots)$, i.e. $a \equiv b((\theta \circ \Phi \circ \theta \circ \ldots)_D)$. Then there exist $c_1, \dots, c_{n-1} \in L$ such that $a \equiv c_1(\theta)$, $c_1 \equiv c_2(\Phi), \dots, c_{n-1} \equiv b(\theta)$ if n is odd or $c_{n-1} \equiv b(\Phi)$ if n is even. Therefore, $a = a \vee a^* \equiv c_1 \vee c_1^*(\theta)$, $c_1 \vee c_1^* \equiv c_2 \vee c_2^*(\Phi), \dots, c_{n-1} \vee c_{n-1}^* \equiv b \vee b^* = b(\theta)$ for n odd or $c_{n-1} \vee c_{n-1}^* \equiv b \vee b^* = b(\Phi)$ for n even. This means $a \equiv b(\theta_D \circ \Phi_D \circ \theta_D \circ \dots)$, because $c_i \vee c_i^* \in D(L)$ for every $i = 1, \dots, n-1$.

Lemma 3.

Let θ and Φ be congruences on a p-algebra L. Let τ denote the relation $\theta \circ \Phi \circ \theta \circ \dots$ (n-times). Then $a \equiv b(\tau)$ and $c \equiv d(\tau)$ imply $a \wedge c \equiv b \wedge d(\tau)$, $a \vee c \equiv b \vee d(\tau)$ and $a^* \equiv b^*(\tau)$.

Proof.

Assume $a \equiv b(\tau)$ and $c \equiv d(\tau)$. Then there exist $a_1, \dots, a_{n-1}, c_1, \dots, c_{n-1} \in L$ such that $a \equiv a_1(\theta)$, $a_1 \equiv a_2(\Phi), \dots, a_{n-1} \equiv b(\theta)$ for n odd or $a_{n-1} \equiv b(\Phi)$ for n even. Similarly, $c \equiv c_1(\theta)$, $c_1 \equiv c_2(\Phi), \dots, c_{n-1} \equiv d(\theta)$ for n odd or $c_{n-1} \equiv d(\Phi)$ for n even. Therefore, $a \wedge c \equiv a_1 \wedge c_1(\theta)$, $a_1 \wedge c_1 \equiv a_2 \wedge c_2(\Phi), \dots, a_{n-1} \wedge c_{n-1} \equiv b \wedge c(\theta)$ for n odd or $a_{n-1} \wedge a_{n-1} \equiv b \wedge d(\Phi)$ for n even. Thus $a \wedge c \equiv b \wedge d(\tau)$. Similarly $a \vee c \equiv b \vee d(\tau)$ and $a^* \equiv b^*(\tau)$, $c^* \equiv d^*(\tau)$.

Theorem 3.

Two congruences θ, Φ on a p-algebra L satisfying the identity $x = x^{**} \wedge (x \vee x^*)$, are n-permutable if and only if their restrictions θ_D, Φ_D on the lattice $D(L)$, are n-permutable.

Proof.

Necessity. Follows from Lemma 2, because $\theta_D \circ \Phi_D \circ \theta_D \circ \dots = (\theta \circ \Phi \circ \theta \circ \dots)_D = (\Phi \circ \theta \circ \Phi \circ \dots)_D = \Phi_D \circ \theta_D \circ \Phi_D \circ \dots$.

Sufficiency. Assume $a \equiv b(\theta \circ \Phi \circ \theta \circ \dots)$ therefore $a^{**} \equiv b^{**}((\theta \circ \Phi \circ \theta \circ \dots)_B)$ and $a \vee a^* \equiv b \vee b^*((\theta \circ \Phi \circ \theta \circ \dots)_D)$ by Lemma 3. Applying Lemma 2 we get: $a^{**} \equiv b^{**}(\theta_B \circ \Phi_B \circ \theta_B \circ \dots)$ and $a \vee a^* \equiv b \vee b^*(\theta_D \circ \Phi_D \circ \theta_D \circ \dots)$. It is known that a Boolean algebra is permutable (i.e. 2-permutable) and consequently, n-permutable. Therefore, $a^{**} \equiv b^{**}(\Phi_B \circ \theta_B \circ \Phi_B \circ \dots)$. By assumption $\theta_D \circ \Phi_D \circ \theta_D \circ \dots = \Phi_D \circ \theta_D \circ \Phi_D \circ \dots$ (n-times), we obtain $a \vee a^* \equiv b \vee b^*(\Phi_D \circ \theta_D \circ \Phi_D \circ \dots)$. Using Lemma 2 we get $a^{**} \equiv b^{**}((\Phi \circ \theta \circ \Phi \circ \dots)_B)$ and $a \vee a^* \equiv b \vee b^*((\Phi \circ \theta \circ \Phi \circ \dots)_D)$. Now, by Lemma 3 and the hypothesis, $a = a^{**} \wedge (a \vee a^*) \equiv b = b^{**} \wedge (b \vee b^*)(\Phi \circ \theta \circ \Phi \circ \dots)$. Thus $\theta \circ \Phi \circ \theta \circ \dots \leqslant \Phi \circ \theta \circ \Phi \circ \dots$ Similarly, one can obtain the converse inclusion.

Since 2-permutablility means permutability, we get:

Corollary 2.

Two congruences θ, Φ on a p-algebra L, satisfying the identity $x = x^{**} \wedge (x \vee x^*)$, are permutable if and only if θ_D, Φ_D are permutable in Con $(D(L))$.

Corollary 3.

A p-algebra L satisfying the identity $x = x^{**} \wedge (x \vee x^*)$ has permutable congruences if $D(L)$ is a relatively complemented lattice.

Proof.

If $D(L)$ is a relatively complemented lattice, then every two congruences are permutable, (see [2]). Thus θ_D, Φ_D permute for every θ, $\Phi \in$ Con (L), which means that θ, Φ are permutable.

Remark.

Let L be a p-algebra satisfying the identity $x = x^{**} \wedge (x \vee x^*)$. The Glivenko congruence $\gamma \in$ Con (L) permuts with every element of Con (L), since $\gamma_B = \Delta_B$ and $\gamma_D = \nabla_D$.

REFERENCES

[1] Grätzer, G.: Lattice theory, first concepts and distributive lattices, Freeman (1971).
[2] Grätzer, G.: Universal Algebra (second edition), Springer-Verlag New York (1979).
[3] Katriňák, T.: On a problem of G. Grätzer, Proc. Amer. Math. Soc. (1) (1976),19—24.
[4] Katriňák, T.: Congruence pairs of p-algebras with a modular frame, Algebra Universalis (8) (1978),205—220.
[5] Katriňák, T.: Essential and strong extentsions of p-algebras, Bull. de la Soc. Roy. (3—4) (1980), 119—124.
[6] Katriňák, T. and Mederly, P.: Construction of p-algebras, Algebra Universalis (to appear).
[7] Schmidt, E. T.: On n-permutable equational classes, Acta Sci. Mat. (Szeged) (33) (1972),29—30.

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SÚHRN

DVE POZNÁMKY O ZVAZE KONGRUENCIÍ p-ALGEBIER

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V práci je daná charakterizácia zväzu kongruencií pomocou kvázimodulárnych p-algebier

РЕЗЮМЕ

ДВЕ ЗАМЕТКИ ОБ РЕШЁТКЕ КОНГРУЕНЦИЙ n-АЛГЕБР

Санаа ЕЛ-Ассар, Братислава

В работе дана характеризация решётки конгруенций с помощью квазимодулярныцх p-алгебр.