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CW DECOMPOSITIONS AND ORIENTABILITY OF *s*-CUBES

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Introduction

In [3] some special factorspaces of an n-dimensional cube I^n, so called s-cubes, were introduced.

In the present paper a *CW* decomposition $\mathcal{F}^n$ of I^n is defined in such a way that for any given s-cube $I^n/(u^1, \ldots, u^n) = I^n/T$ the equivalence relation T is a celluar one on the *CW* space $(I^n, \mathcal{F}^n)$. This enables to construct a *CW* decomposition $\mathcal{F}^n/T$ of I^n/T and to introduce a *CW* structure to the s-cube X. In the case when the s-cube X is a manifold, computing then n-th homology group we give a simple condition to decide whether X is orientable or not.

1. Notation and basic definitions

Let $n \geqq 1$ be an integer. We shall use the following notation:

$N_n = \{1, 2, \ldots, n\}$

$I^n = \{x \in R^n;\ |x_i| \leqq 1,\ i \in N_n\}$ is the n-dimensional cube, $I^0 = \{0\}$

∂I^n is the boundary of I^n

$B^n = \{x \in R^n;\ \|x\| \leqq 1\}$ is the n-dimensional ball

$S^{n-1} = \partial B^n$ is the $(n-1)$-dimensional sphere

$J_i^n = \{x \in I^n;\ |x_i| = 1\}$ is the i-th double-face of the cube I^n

$s_i\colon \partial I^n \to \partial I^n$, $x \mapsto (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$ is the symmetry of ∂I^n with respect to the hyperplane $x_i = 0$; $i \in N_n$

G is the subgroup of the group of all transformations of ∂I^n generated by the set $\{s_i;\ i \in N_n\}$

Group G is abelian, because $G \cong (Z_2)^n$. Each $u \in G$, $u \neq 1$, is the product of mutually different transformations $s_{i_1}, \ldots, s_{i_k}$ and it can be uniquely written in the form $s_{i_1 i_2 \ldots i_k} = s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_k}$, where $i_1 < i_2 < \ldots < i_k$.

Definition 1. Let $u^1, \ldots, u^n \in G$ be given. An s-cube $I^n/(u^1, u^2, \ldots, u^n)$ is a factorspace I^n/T where T is the equivalence relation on I^n defined as follows:

$$x\ T\ y \Leftrightarrow x = y \text{ or there are numbers } i_1, \ldots, i_k \in N_n \text{ such that}$$

$$x, y \in \bigcap_{j=1}^{k} J_{i_j} \text{ and } x = u^{i_1} \circ \ldots \circ u^{i_k}(y)\,.$$

Example. For $n=2$ we have: $I^2/(s_1, s_1) \approx S^2$, $I^2/(s_1, s_2) \approx T^2$, $I^2/(s_1, id) \approx$-$S^1 \times I$, $I^2/(s_1, s_{12}) \approx Kb, \ldots$, where T^2 is the 2-dimensional torus and Kb is the Klein bottle.

2. *CW* decomposition $\mathcal{F}^n$ pf I^n

In this section we shall construct a *CW* decomposition $\mathcal{F}^n$ of the cube I^n in such a way that for any s-cube $I^n/(u^1, \ldots, u^n) = I^n/T$ the equivalence relation T will be celluar on the *CW* space $(I^n, \mathcal{F}^n)$.

In the construction of *CW* decomposition $\mathcal{F}^n$ we shall identify $I^n \equiv B^n$ via the standard homeomorphism v: $(I^n, \partial I^n) \to (B^n, S^{n-1})$ defined by the radial projection, see [1], page 55.

CW decomposition $\mathscr{E}^1$ of I^1.

The set $\mathscr{E}^1$ consists of 5 cells: three 0-cells e^0_{-1}, e^0_0, e^0_1 and two 1-cells e^1_{-1}, e^1_1. The characteristic maps are:

$$f^0_{-1}, f^0_0, f^0_1 : I^0 \to I^1, f^0_{-1}(0) = -1, f^0_0(0) = 0, f^0_1(0) = 1\,,$$

$$f^1_{-1}, f^1_1 : I \to I, f^1_{-1}(x) = \frac{-x-1}{2}, f^1_1(x) = \frac{x+1}{2}\,.$$

CW decomposition $\mathscr{E}^n$ of I^n.

The *CW* decomposition $\mathscr{E}^n$ of I^n is the product *CW* decomposition $\mathscr{E}^n = \mathscr{E}^1 \times \ldots \times \mathscr{E}^1$ (n times). We shall use the following notation:

$\mathscr{E}^n = \{e(q,(p(q));\ q \in \{0, 1\}^n,\ p\colon \{0, 1\}^n \to \{-1, 0, 1\}^n$ is a map such that $p_i(q) \in \{-1, 0, 1\}$ for $q_i = 0$ and $p_i(q) \in \{-1, 1\}$ for $q_i = 1\}$.

The dimension of the cell $e(q, p(q))$ is $\sum_{i=1}^{n} q_i$ and the characteristic map of a k-dimensional cell $e(q(p(q))$ is the map $f(q, p(q))\colon I^k \to I^n$, $(t_1, \ldots, t_k) \mapsto (x_1, \ldots, x_n)$ where $x_j = p_j(q)$ for $q_j = 0$ and $x_j = f^1_{p_j(q)}(t_r)$ for $q_j = 1$, $r = \sum_{i=1}^{j} q_i$.

The *CW* decomposition $\mathscr{E}^n$ of I^n is such that for any given s-cube I^n-$/(u^1, \ldots, u^n) = I^n/T$ the relation T is the celluar equivalence relation[1)] on the *CW*

[1)] See [2], page 32.

space $(I^n, \mathscr{E}^n)$. But for the future computation the number of cells of CW decomposition $\mathscr{E}^n$ is rather high, card $\mathscr{E}^n = 5^n$. Since the glueing operation by which the cube I^n is turned to the s-cube X deals only with the boundary ∂I^n of I^n, we can simplify the CW decomposition $\mathscr{E}^n$ by exchanging all interior cells with only one new n-dimensional cell e^n.

CW decomposition $\mathscr{F}^n$ of I^n.

Definition 2. A cell $e(q, p(q)) \in \mathscr{E}^n$ is called a boundary cell if there is $i \in N_n$ such that $q_i = 0$ and $p_i(q) = \pm 1$.

Denoting by $\mathscr{B}^n$ the set of all boundary cells in $\mathscr{E}^n$ we have the following

Proposition 1. The set $\mathscr{F}^n = \mathscr{B}^n \cup \{e^n\}$ of cells, where e^n is the n-dimensional cell in I^n with the characteristic map id: $I^n \to I^n$, is the CW decomposition of I^n. The number of cells in $\mathscr{F}^n$ is $5^n - 3^n + 1$. A cell $e(q, p(q)) \in \mathscr{F}^n$ is contained in the double-face J_i of I^n if and only if $q_i = 0$ and $p_i(q) = \pm 1$.

3. CW decomposition of s-cube $I^n/(u^1, \ldots, u^n)$

Let $X = I^n/(u^1, \ldots, u^n) = I^n/T$ be an s-cube and $\{i_1, \ldots, i_k\}$ a nonempty subset of N_n. We introduce the following notation:

$h_n\colon I^n \to I^n/T$ is the canonical projection

$J(\{i_1, \ldots, i_k\}) = \bigcap_{j=1}^{k} J_{i_j}$

$G(\{i_1, \ldots, i_k\})$ is the subgroup of G generated by the set $\{u^{i_1}, \ldots, u^{i_k}\}$

$\Psi\colon G \to \{-1, 1\}^n$, $s_i \mapsto (x_1, \ldots, x_n)$ where $x_i = -1$ and $x_j = 1$ for $j \neq i$

$S(e(q, p(q))) = \{k \in N_n, e(q, p(q)) \subset J_k^n\}$, $e(q, p(q)) \in \mathscr{F}^n$.

Now a CW decomposition of the s-cube X will be constructed as a factor decomposition of $\mathscr{F}^n$.

Let $e(q, p(q))$ be a cell from $\mathscr{F}^n$ and $u \in G(S(e(q, p(q))))$. Denote by $e(p, u \circ p(q))$ the cell $e(q, (u_1 p_1(q), \ldots, u_n p_n(q)))$, where $(u_1, \ldots, u_n) = \Psi(u)$. Then

a) $h_n^{-1} \circ h_n(e(q, p(q))) = \bigcup \{e(q, u \circ p(q));\, u \in G(S(e(q, p(q))))\}$.

b) $v|_{e(q, p(q))}\colon e(q, p(q)) \to e(q, v \circ p(q))$ is a homeomorphism for any $v \in G(S(e(q, p(q))))$.

c) $h_n|_{e(q, p(q))}\colon e(q, p(q)) \to h_n(e(q, p(q)))$ is a homeomorphism

With regard to a), b), c), we have

Theorem 1. The equivalence relation T on I^n is celluar with respect to CW decomposition $\mathscr{F}^n$ of I^n.

Corollary. The set $\mathscr{F}^n/T = \{h_n(e);\, e \in \mathscr{F}^n\}$ is a CW decomposition of I^n/T.

The characteristic maps for cells from $\mathscr{F}^n/T$ can be chosen in this way: For every $e \in \mathscr{F}^n/T$ we find one cell $\tilde{e} \in \mathscr{F}^n$ such that $h_n(\tilde{e}) = e$. Denoting by f the characteristic map of $\tilde{e}$, the map $h_n \circ f$ is characteristic for e.

4. Orientability of s-cubes

In this section we shall deal with the problem of orientability of those s-cubes which are manifolds. To decide whether a given s-cube is orientable or not, it is sufficient to compute its n-th homology group over Z.

Let $X = I^n/(u^1, \ldots, u^n) = I^n/T$ be an s-cube and let $\mathcal{F}^n/T$ be the CW decomposition of I^n/T introduced in part 3. We recall that in this decomposition there is only one n-cell, namely $h_n(e^n)$. Denoting by CX the cell chain complex of the CW space $(X, \mathcal{F}^n/T)$ (and by ∂ the boundary operator), we have $H_nX = \operatorname{Ker} \partial_n$, because $\operatorname{Im} \partial_{n+1} = 0$. To describe the boundary operator $\partial_n\colon C_nX \to C_{n-1}X$, we compute all incidence numbers $[a:b]$, where $a = h_n(e^n)$ and $b \in C_{n-1}X$ is an $(n-1)$-dimensional cell of $(X, \mathcal{F}^n/T)$.

Lemma 1. Let $b = h_n(e(q, p(q))$ be an $(n-1)$-cell in $\mathcal{F}^n/T$ such that $e(q, p(q)) \subset J_i^n$ for some $i \in N_n$. Suppose that $u^i = \mathrm{id} \circ s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_k}$ for $i_1 < i_2 < \ldots < i_k$, $k \geqq 0$. Then

a) $u^i = \mathrm{id} \Rightarrow [a:b] = \pm 1$

b) $u^i \neq id \Rightarrow [a:b] = 0$ for k odd, $[a:b] = \pm 2$ for k even.

Proof: Let us denote by Φ^a and φ^a the characteristic and the attaching maps of the cell a, similarly Φ^b, φ^b for the cell b. According to [1], Corollary 6.11, the incidence number $[a:b]$ coincides with the local degree of the map

$$F = (\Phi^b)^{-1} \circ \varphi^a\colon (\varphi^a)^{-1} b \to I^{n-1}$$

over any point $Q \in \operatorname{int}(I^{n-1})$. We are going to discuss cases a), b).

a) In this case the set $(\varphi^a)^{-1}Q$ consists of one point P. Since F is a homeomorphism, it is $[a:b] = \pm 1$.

b) We have $(\varphi^a)^{-1}Q = \{P_1, P_2\}$ and $\deg_Q F = \mu(P_1) + \mu(P_2) = \mu(P_1) + \deg u^i \mu(P_1) = \mu(P_1)\,(1 + (-1)^k)$. Since F is a local homeomorphism, it is $\mu(P_1) = \pm 1$ and $\deg_Q F = \pm 2$ for k even, $\deg_Q F = 0$ for k odd.

Let $i \in N_n$ and let $M_i \subset N_n$ be such a set that $u^i = \prod_{j \in M_i} s_j$. Put $M_i = \emptyset$ if $u^i = \mathrm{id}$. Denote $c_i = \operatorname{card} M_i$, $c = c_1 c_2 \ldots c_n$.

Theorem 2. $H_nX = 0$ for c even, $H_nX = Z$ for c odd.

Proof. Applying Lemma 1 we get: $\operatorname{Ker} \partial_n = Z$ if and only if c_i is odd for every $i \in N_n$. The assertion follows.

Corollary: Let $X = I^n/(u^1, \ldots, u^n)$ be a manifold. Then X is orientable if and only if c is odd.

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SÚHRN

CW ROZKLADY A ORIENTOVATEĽNOSŤ s-KOCIEK

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V práci je definovaný taký *CW* rozklad $\mathcal{F}^n$ n-rozmernej kocky I^n, že pre každú s-kocku $X = I^n/(u^1, \ldots, u^n) = I^n/T$ je ekvivalencia T bunečná relácia ekvivalencie na $(I^n, \mathcal{F}^n)$. To umožňuje zaviesť štruktúru *CW* komplexu na ľubovoľnú s-kocku X. Pre také s-kocky, ktoré sú variety, je v práci daná nutná a postačujúca podmienka ich orientovateľnosti.

РЕЗЮМЕ

КЛЕТОЧНЫЕ РАЗБИЕНИЯ И ОРИЕНТИРУЕМОСТЬ s-КУБОВ

Милош Божек—Йозеф Тварожек, Братислава

В работе определено клеточное разбиение $\mathcal{F}^n$ n-мерного куба I^n и показано, что для любого s-куба $X = I^n/(u^1, \ldots, u^n)$ соответствующее отношение эквивалентности T на I^n является клеточным отношением на $(I^n, \mathcal{F}^n)$. Разбиение $\mathcal{F}^n$ позволяет ввести клеточную структуру для любого s-куба. С ее помощью доказывается необходимое и достаточное условие для ориентируемости любого s-куба, являющегося многообразием.