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MAXIMAL SIZES OF PLANAR DIGRAPHS WITHOUT TRANSITIVE EDGES

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The terminology in this paper as well as denotation is based on [7] except for that given here. An edge uv of a digraph G is called transitive, if there is a directed $u-v$-path in $G-uv$. A minimal digraph is a digraph containing no transitive edge.

One might ask to find a spanning subdigraph of G without transitive edges and with the least possible size (i.e. the least possible number of edges). However, as it is shown in [10], this problem is NP-complete even for planar digraphs with very restricted degrees. Thus, it is a natural question to determine at least extremal sizes of minimal digraphs. (Questions of this kind are very frequent in graph theory; there is even a book [2] on extremal problems in graph theory.) It is well known [3] that any minimal strong digraph contains vertices with both the indegree and outdegree equal to one, and there are at least two such vertices (see [1]), which is the best possible bound on the number of such vertices. The same holds for minimal strong blocks [5]. In [8] it is proved that any minimal strongly k-connected digraph contains a vertex of indegree k or a vertex of outdegree k. As follows from [4], [6], [9] any minimal strong digraph with p vertices has at most $2p-2$ edges, and the bound $2p-3$ is given in [5] for minimal strong blocks. Both these bounds are sharp. As shown in [6] any minimal digraph of order p has at most $\max\{2p-2, [p^2/4]\}$ edges. In this note we give maximal sizes of minimal digraphs for various classes of planar digraphs. They are slightly different from the general bounds.

Theorem 1. Let G be a minimal, acyclic and planar (p, q)-digraph with $p \geq 3$. Then $q \leq 2p-4$.

Proof. Let H be the underlying graph of G. (Since G is acyclic, H does not contain multiple edges.) The length of every cycle in H is at least four (any orientation of a 3-cycle in H gives either directed 3-cycle or a transitive edge in G).

Hence, the length of a boundary of any face of H is at least four, and (from Euler's equation) we have: $q \leq 2p-4$.

Remark 1. If G is a minimal acyclic and planar digraph and the length of a boundary of any face of G is exactly four, then $q=2p-4$. The following construction *(Fig. 1)* gives a sequence of such digraphs for $p=4k$, where k is an integer, $k \geq 1$. Hence, Theorem 1 is sharp.

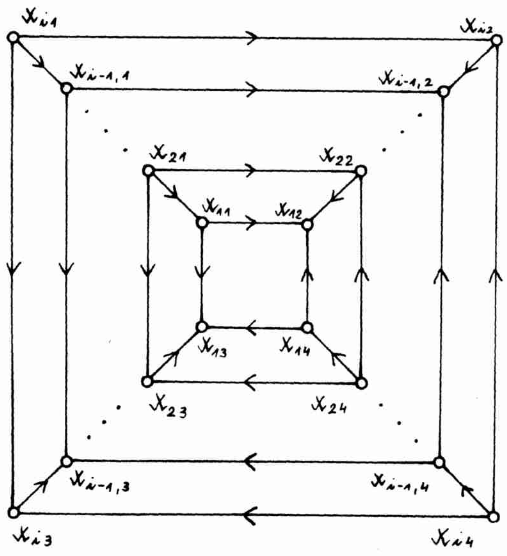

Fig. 1

We define a digraph G_i for an arbitrary natural number i recursively as follows:

$$\begin{aligned}
G_1: \quad & V(G_1) = \{x_{11}, x_{12}, x_{13}, x_{14}\} \\
& E(G_1) = \{x_{11}x_{12}, x_{11}x_{13}, x_{14}x_{12}, x_{14}x_{13}\} \\
G_i(i\geq 2): \quad & V(G_i) = V(G_{i-1}) \cup \{x_{i1}, x_{i2}, x_{i3}, x_{i4}\} \\
& E(G_i) = E(G_{i-1}) \cup \{x_{ij}x_{i-1,j} \mid j=1, 2, 3, 4\} \cup \\
& \cup \{x_{i1}x_{i2}, x_{i1}x_{i3}, x_{i4}x_{i3}, x_{i4}x_{i2}\}
\end{aligned}$$

It is easy to verify that G_i is a minimal acyclic and planar $(4i, 8i-4)$-digraph.

Theorem 2. Let G be a minimal planar (p, q)-digraph. Then $q \leq 2(p-1)$.

Proof. (By induction.)

The statement is obvious for $p \leq 3$.

Let G be a minimal planar (p, q)-digraph. We will distinguish the following possible cases:

a) If G is acyclic, then from Theorem 1 it immediately follows that

$$q \leq 2p - 4 < 2(p-1)\,. \tag{1}$$

b) If G contains no directed cycle of the length less or equal to 3, then the length of a boundary of any face in G is at least 4 and (as it follows from the proof of Theorem 1) the inequality (1) holds.

c) Let G contains a directed t-cycle with $t \leq 3$. Assume that G is embedded in a plane, i.e. G is a plane digraph. We construct a new digraph H by the removal of all vertices and edges of the t-cycle and the addition of a new vertex w adjacent to those vertices to which at least one of the vertices of the t-cycle was adjacent and adjacent from those vertices from which at least one of the vertices of t-cycle was adjacent. The illustrations of this operation for $t=2$ and $t=3$ are given in *Fig. 2*

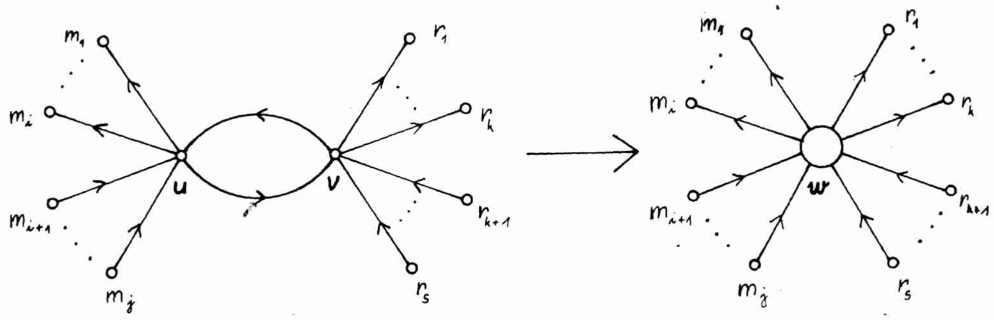

Fig. 2

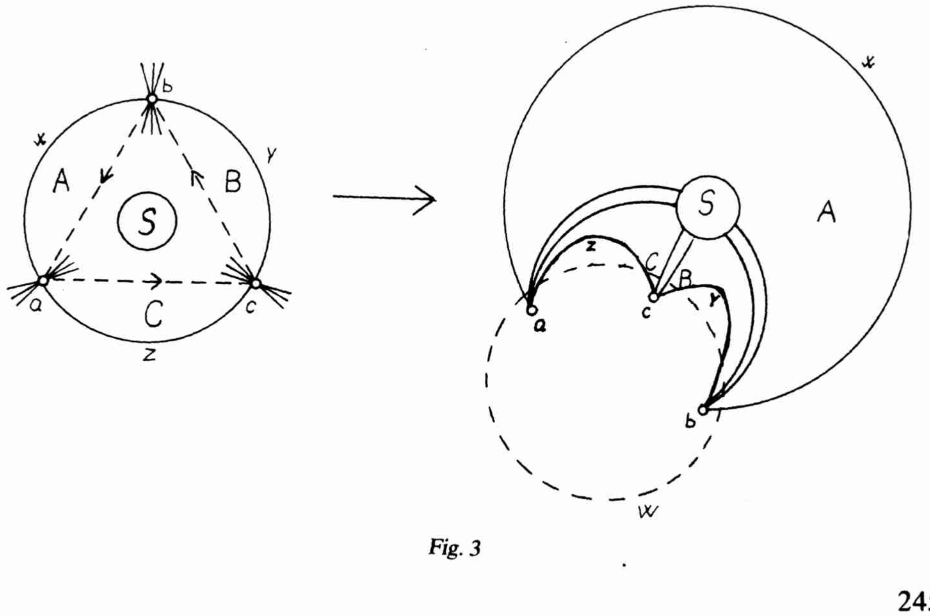

Fig. 3

and *Fig. 3* respectively. A, B, C denote the faces lying out from the 3-cycle and containing the edges ba, cb, ac respectively.

From these illustrations it is easy to see that the operation (mentioned above) saves the planarity of the digraph. Hence H is a planar digraph.

In the following we shall show the minimality of H. Let at first $t=2$, and xy be a transitive edge in H. Then there exists a directed $(x-y)$-path $x \equiv n_1, n_2, \ldots, n_k \equiv y$ in $H(k \geqslant 1$ is an integer), containing the vertex w, but not containing the edge xy. Let $w \equiv n_i$, where $i \in \langle 1, k \rangle$. Then there would be a directed $(x-y)$-path $x \equiv n_1, n_2, \ldots, n_{i-1}, u, v, n_{i+1}, \ldots, n_k \equiv y$ in G (or $u, v, n_2, \ldots, n_k \equiv y$ for $i=1$), which is contradiction.

Let $t=3$, and xy be a transitive edge in H. Then there exists a directed $(x-y)$-*path* P: $x \equiv m_1, m_2, \ldots, m_k \equiv y$ containing w, but not containing the edge xy. Analogously to the previous case it can be shown that the transitivity of xy in H implies the transitivity of the same edge in G, too. To prove this, it is sufficient to take into account all possible replacements of w by a 3-cycle in P. Thus H is minimal.

For $t=2$ the digraph H is a $(p-1, q-2)$-digraph. From the induction hypothesis it follows that:

$$q-2 \leqslant 2(p-1)-2,$$

i.e.

$$q \leqslant 2p-2.$$

For $t=3$ the digraph H is a $(p-2, q-3)$-digraph. In this case the induction hypothesis gives:

$$q-3 \leqslant 2[(p-2)-1]$$

hence

$$q \leqslant 2p-3 < 2p-2.$$

This completes the proof.

Remark 2. Let F be a tree with p vertices. We construct a digraph G by replacement of any edge uv by two directed edges uv and vu. Obviously, G is minimal and planar $(p, 2p-2)$-digraph. This construction gives a minimal planar (p, q)-digraph for any integer $p \geqslant 2$, with $q=2(p-1)$.

Theorem 3. Let G be an outerplanar minimal and acyclic (p, q)-digraph $(p \geqslant 2)$. Then

$$q \leqslant \frac{3p-4}{2}. \tag{2}$$

Proof. Let H be underlying graph of G. It can be embedded in the plane so that all its vertices lie on the same (exterior) face. Since G is acyclic and minimal, H

has no multiple edge and does not contain any 3-cycle. Thus the length of a boundary of any interior face in H is at least four. Then

$$2q \geqslant p + 4r\,, \tag{3}$$

where r denotes the number of all the interior faces of H. From Euler's equation we have

$$r + 1 + p - q = 2\,,$$

that is

$$r = q - p + 1\,. \tag{4}$$

Substituting (4) into (3) gives:

$$2q \geqslant p + 4q - 4p + 4$$

hence

$$q \leqslant \frac{3p-4}{2}\,.$$

This completes the proof.

Remark 3. Let us define a $(4i,\, 6i-2)$-digraph as follows *(Fig. 4)*:

$$\begin{aligned}
G_1: \quad & V(G_1) = \{x_{11}, x_{12}, x_{13}, x_{14}\} \\
& E(G_1) = \{x_{12}x_{11}, x_{11}x_{14}, x_{12}x_{13}, x_{13}x_{14}\} \\
G_i(i \geqslant 2): \quad & V(G_i) = V(G_{i-1}) \cup \{x_{i1}, x_{i2}, x_{i3}, x_{i4}\} \\
& E(G_i) = E(G_{i-1}) \cup \{x_{i-1,2}, x_{i2}, x_{i-1,3}, x_{i1}, \\
& \qquad x_{i2}x_{i1}, x_{i2}x_{i3}, x_{i1}x_{i4}, x_{i3}x_{i4}\}
\end{aligned}$$

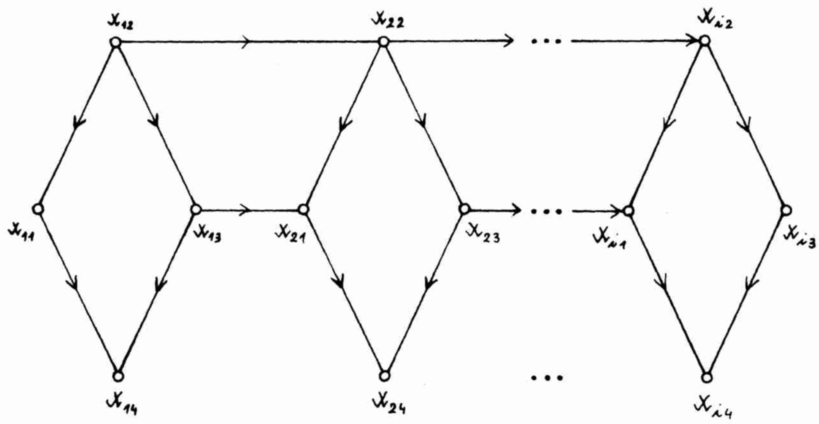

Fig. 4

All the required properties of G_i can be easily verified. Hence for any natural i G_i is an example of an outerplanar minimal and acyclic (p, q)-digraph with $p = 4i$ and $q = \frac{3p-4}{2}$.

Remark 4. Since (as follows from Remark 2) for any natural p there is an outerplanar and minimal (p, q)-digraph with $q = 2(p-1)$, the estimation for outerplanar digraphs is in general the same as for planar digraphs.

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РЕЗЮМЕ

МАКСИМАЛЬНАЯ РАЗМЕРНОСТЬ ПЛАНАРНЫХ ДИГРАФОВ БЕЗ ТРАНЗИТИВНЫХ РЕБЕР

Петер Кыш—Алойз Ваврух, Братислава

Авторы в работе показывают верхные ограничения количества ребер минимальных диграфов для различных классов планарных графов.

SÚHRN

MAXIMÁLNY ROZMER PLANÁRNYCH DIGRAFOV BEZ TRANZITÍVNYCH HRÁN

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Autori v práci podávajú horné ohraničenia počtu hrán minimálnych digrafov pre rôzne triedy planárnych digrafov.