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REMARKS ON DOMATIC NUMBER

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In this article we shall investigate two questions on domatic number of graphs. The first one is concerned with questions introduced in [2], where Harary and Kabell investigated the change of a given graphical parameter $\pi(G)$ in a specified direction by taking a finite sequence of positive integers which is monotone nondecreasing $a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n$ (or monotone nonincreasing $a_0 \geqslant a_1 \geqslant \ldots \geqslant a_n$) and constructing a graph G with distinguished points $v_1, v_2, \ldots, v_n$ such that $\pi(G) = a_0$ and $\pi(G - v_1 - v_2 - \ldots - v_i) = a_i$ for $1 \leqslant i \leqslant n$. They have shown the existence of such graphs for some graphical parameters such as the point and line connectivity, the minimum and maximum degree, the diameter, the chromatic number, etc. In this paper we shall investigate this problem for the domatic number of a graph. In the second part we deal with a conjecture of Zelinka [3] on domatic numbers of cubes. We give a counterexample to his conjecture for one special case. All the terminology is taken from [1] except for the given here.

1. Let $G = (V, E)$ be an undirected graph without loops and multiple lines. A subset $R \subset V$ is called a dominating set in G if to each point $x \in V - R$ there exists a point $y \in R$ adjacent to x. A domatic partition of G is a partition of V, all classes of which are dominating sets in G. The maximal number of classes of a domatic partition of G is called the domatic number [4] of G and denoted by $d(G)$. Given two subsets $A \subset V$ $B \subset V$ we will say that A is covered by B if for every point $x \in A$ there exists an adjacent point $y \in B$.

Theorem 1. For every sequence of positive integers which is monotone nondecreasing, $a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n$, there exists a graph $G = (V, E)$ with distinguished points $v_1, v_2, \ldots, v_n$ such that $d(G) = a_0$ and $d(G - v_1 - v_2 - \ldots - v_i) = a_i$ for $1 \leqslant i \leqslant n$.

Proof. Let us consider a graph G consisting of a complete graph K_{a_n} with the point set $X = \{x_1, x_2, \ldots, x_{a_0-1}, \ldots, x_{a_n}\}$ and of other points $v_1, v_2, \ldots, v_n$, where v_i is adjacent to $a_{i-1} - 1$ first points of X. Put $G_i = G - v_1 - v_2 - \ldots - v_i$ for $i = 1, \ldots, n$ and $G_0 = G$. As it follows from [4] $d(G_i) \leqslant \delta_i + 1$, where $\delta_i = a_i - 1$ is the minimum

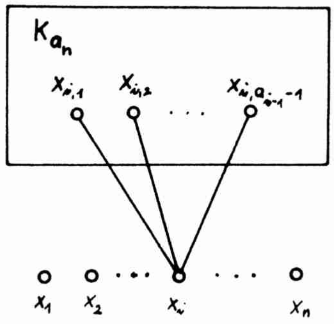

Fig. 1

degree of G_i. Thus $d(G_i) \leq a_i$. To prove that $d(G_i) = a_i$ we construct the following partition of V:

$$A_k(i) = \{x_k\}, \quad k = 1, 2, \ldots, a_i - 1$$

$$A_{a_i}(i) = V(G) - \{x_1, \ldots, x_{a_i - 1}\}$$

Since all the sets $A_k(i)$, $k = 1, \ldots, a_i$ are dominating in G_i we have $d(G_i) = a_i$. This completes the proof.

One can prove very easily that by deleting a point (or a line) the domatic number reduces at most by one.

Theorem 2. For every sequence of integers $a_0 \geqslant a_1 \geqslant \ldots \geqslant a_n$ with $a_i - a_{i+1} \leqslant 1$ for $i = 0, 1, \ldots, n-1$ there exists a graph $G = (V, E)$ with distinguished points v_1, $v_2, \ldots, v_n$ such that $d(G) = a_0$ and $d(G - v_1 - v_2 - \ldots - v_i) = a_i$.

Proof. The number of pairs (a_{i+1}, a_i) for which $a_{i+1} = a_i$ is denoted by k. Let K_{a_0} be the complete graph, $t_0 \in V(K_{a_0})$. Let us construct a graph G as follows *(Fig. 2)*:

$$V(G) = V(K_{a_0}) \cup \{t_i\}_1^k \quad t_i \notin V(K_{a_0})$$

$$E(G) = E(K_{a_0}) \cup \{t_i x \mid t_0 x \in E(K_{a_0}), \quad i = 1, 2, \ldots, k\}\ .$$

It defined the sequence of vertices $v_1, v_2, \ldots, v_n$ in the following manner:

if $a_i - a_{i+1} = 0$ we take v_i from $\{t_1, t_2, \ldots, t_k\}$ and otherwise we take $v_i \in V(K_{a_0})$, $v_i \not\equiv t_0$. One can easily see that the assertions of the theorem hold. This completes the proof.

Theorem 3. For every sequence of integers $a_0 \geqslant a_1 \geqq \ldots \geqslant a_n$ which is monotone nonincreasing and $a_i - a_{i+1} \leqslant 1$ for $i = 0, 1, \ldots, n-1$ there exists a graph $G = (V, E)$ with distinguished lines $e_1, e_2, \ldots, e_n$ such that $d(G) = a_0$ and $d(G - e_1 - e_2 - \ldots - e_i) = a_i$.

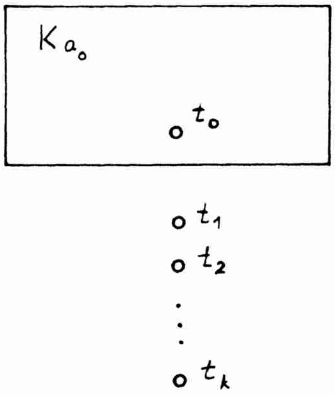

Fig. 2

Proof. Let us construct G by taking K_N with $N \geqslant n + a_0$ and a point x which is adjacent to any $a_0 - 1$ points $t_1, t_2, \ldots, t_{a_0-1}$ of the complete graph K_N *(Fig. 3)*.

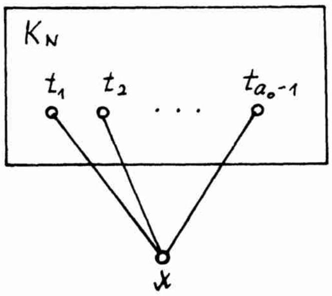

Fig. 3

Obviously $d(G) \leqslant a_0$. We can sonstruct a partition $\{t_i\}_1^{a_0-1}$ and $\{x\} \cup V(K_N) - \{t_1, t_2, \ldots, t_{a_0-1}\}$ of V into a_0 dominating sets. Thus $d(G) = a_0$. We define the required sequence of lines as follows: if $a_{i+1} = a_i$ then we put $e_i = uv$, where u, $v \in V(K_N)$, otherwise we take $e_i = xt_s$, $1 \leqslant s \leqslant a_0 - 1$. This sequence satisfies the assertions of the theorem. This completes the proof.

2. Let k be a positive integer. In [3] Zelinka proved that the graph of the cube of the dimension $2^k - 1$ and the graph of the cube of the dimension 2^k both have the domatic number 2^k. Let Q_n be the graph of n-dimensional cube, where n is a positive integer such that $n + 1$ is not a power of 2. Author conjectured that in this case it holds: $d(Q_n) = n$. The next assertion shows that the conjecture does not hold for $n = 5$.

Assertion 1. $d(Q_5)=4$.

Proof. Let us consider a partition of $V(Q_5)$ into four sets $A=\{a_i\}_1^8$, $B=\{b_i\}_1^8$, $C=\{c_i\}_1^8$, $D=\{d_i\}_1^8$ with the property that the induced subgraphs $\langle A\rangle$, $\langle B\rangle$, $\langle C\rangle$ and $\langle D\rangle$ are 3-dimensional cubes and $E(Q_5)=E(\langle A\rangle)\cup E\langle B\rangle)\cup E(\langle C\rangle)\cup E(\langle D\rangle)\cup\{a_ib_i\}_1^8 \cup \{b_ic_i\}_1^8 \cup \{c_id_i\}_1^8 \cup \{d_ia_i\}_1^8$.

Let R be a dominating set in Q_5. We shall show that $|R|>6$. In order to obtain a contradiction let us assume that $|R|\leq 6$. Then the intersection of R with each of the sets A, B, C, and D is nonempty and there exists at least one of these sets for which this intersection contains exactly one point. With respect to the symmetry we can assume $A\cap R=\{a_1\}$. Since $N(a_1)=\{a_2, a_5, a_4\}$ (see *Fig.* 4), the points a_6, a_7, a_8, a_3 must be covered by the points of $B\cup D$. Thus $S=(B\cup D)\cap R=\{b_6, b_7, b_8, b_3, d_6, d_7, d_8, d_3\}$ and, at the same time, $|(B\cup D)\cap R|=4$.

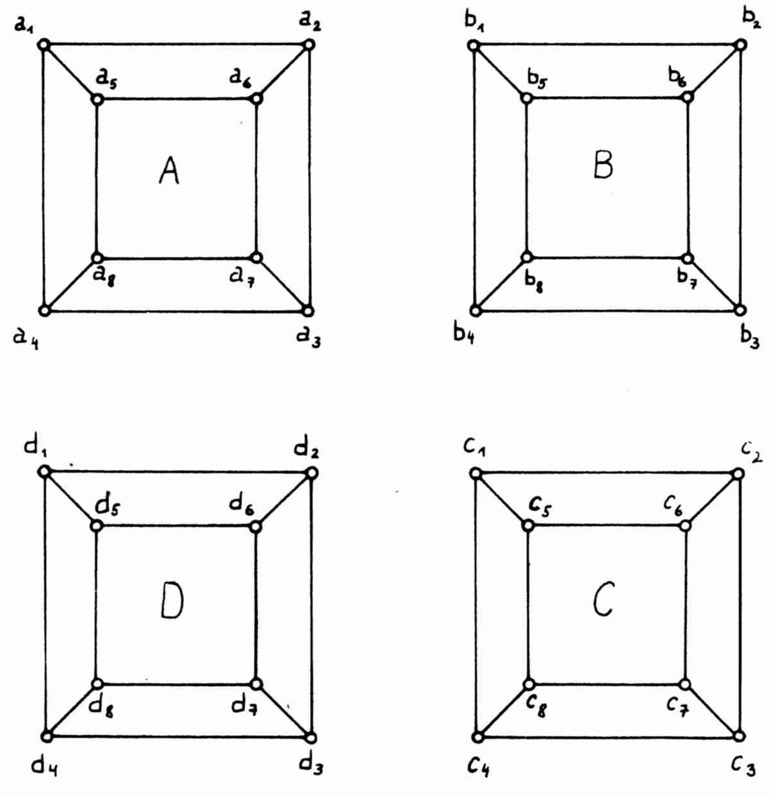

Fig. 4

If $|B\cap R|>2$, ($|D\cap R|>2$) then there exists such a point in D, (B), which is not covered by the points of R, respectively. Hence we have $|B\cap R|=|D\cap R|=2$. No choice of the two points from $B\cap R$ and two points from $D\cap R$ can cover the points c_1, c_4, c_5, c_2. Since $|R| \leq 6$ we have $c_1 \in R$. One can easily verify that for any two points from $B \cap S$ belonging to R there exists a point from B which is not covered by R. Hence $|R| > 6$ and $d(G) \leq 4$. Since V can be partitioned into four dominating sets we have $d(G) = 4$. This completes the proof.

We suppose that the conjecture does not hold also for $n > 5$.

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SÚHRN

POZNÁMKY O DOMATICKOM ČÍSLE

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Pre danú monotónnu postupnosť prirodzených čísel $\{a_k\}_0^n$ je v práci vyšetrovaná otázka existencie grafu G s postupnosťou vrcholov $\{v_i\}_1^n$ $(v_i \neq v_j$ pre $i \neq j)$, pre ktorý platí: $d(G) = a_0$, $d(G - v_1 - v_2 - \ldots - v_i) = a_i$ $(1 \leq i \leq n)$. V druhej časti práce autor na príklade Q_5 ukazuje, že hypotéza ([3]): $d(Q_n) = n$ pre n ktoré nie je mocninou 2, neplatí (Q_n označuje graf n-rozmernej kocky).

РЕЗЮМЕ

ЗАМЕТКИ О ДОМАТИЧЕСКОМ ЧИСЛЕ

Петер Кыш, Братислава

Для данной монотонной последовательности $\{a_k\}_0^n$ в работе рассматривается вопрос существования графа G с последовательностью вершин $\{V_i\}_1^n$ $(v_i \neq v_j$ для $i \neq j)$, для которого:

$$d(G) = a_0, \quad d(G - v_1 - v_2 - \ldots - v_i) = a_i, \quad (1 \leq i \leq n).$$

Во второй части работы автор показывает, что для Q_5 неправильна гипотеза ([3]), согласно которой $d(Q_n) = n$ для n не являющегося степень двух.