UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XLVI—XLVII—1985

REMARKS ON DIAMETERS OF ORIENTATIONS OF GRAPHS

JÁN PLESNÍK, Bratislava

The aim of this paper is to give remarks on some problems concerning orientations of graphs or mixed graphs. In the first section some basic questions are mentioned (acyclic, strong, and other orientations). The second section is devoted to minimum diameter orientations of graphs. We present the smallest graph of diameter 2 whose all strong orientations have diameter at least 6 (which is the maximum possible value as shown by Chvátal and Thomassen [3]). Further we give certain bounds on the least possible diameter of an orientation of a complete k-partite graph or a cube. Thus a result of Boesch and Tindell [2] is extended. In the last (third) section we extend some results of Chvátal and Thomassen [3] on orientations of multigraphs to weighted multigraphs. Bounds on the diameter (radius) of orientations are expressed in terms of the original diameter (radius) and lower and upper bounds on the edge lengths.

Our terminology is based on [1] or [4]. In fact a translation of that in [9] is used. A multimigraph (mixed multigraph) contains no loops but multiple undirected or directed edges between two vertices are allowed.

1. On acyclic, strong, and some other special orientations

It is clear that *every multigraph has an acyclic orientation.* Indeed, supposing that the multigraph has n vertices $v_1, v_2, \ldots, v_n$, it is sufficient to orient an edge joining v_i and v_j from v_i to v_j when $i<j$ and from v_j to v_i otherwise. The same argument can be used to prove that: *A multimigraph G has an acyclic orientation if and only if the subgraph G_D of G consisting of all vertices and all directed edges is acyclic* (Lambin and Tanaev [5]). Namely, supposing that G_D is acyclic, one can denote its vertices by $v_1, v_2, \ldots, v_n$ in such a way that every edge of G_D goes from v_i

to v_j if $i<j$ and conversely otherwise. Then one can easily direct the undirected edges from smaller to greater indices and obtain an acyclic orientation of G.

There is a well known result due to Robbins [10]: *A multigraph has a strong orientation iff it is connected and bridgeless.* Robbins also showed an application of this result to a traffic control problem.

An algorithmic proof of the Robbins theorem based on the depth-first search procedure can be found in [11]. This problem can be slightly generalized: A strong multimigraph is looked upon as being the map of a city; its lines are streets or rooads. When is it possible to introduce a traffic direction in a prescribed street such that any two points are mutually reachable? The following result of Boesch and Tindell [2] solves the problem: *An undirected edge e of a strong multimigraph G can be directed to produce another strong multimigraph if and only if e is not a bridge of G* (i.e. $G-e$ is at least weakly connected multimigraph). The necessity is clear. A proof of the sufficiency simpler than that in [2] is the following: The condensation (see e.g. [4]) of $G-e$ must be a directed path $S_1S_2 \dots S_k$ $(k \geq 1)$ and in $G-e$ one end of e, say u, belongs to the strong component S_1 and the other, say v, to S_k. Directing e from v to u, we obtain the desired multimigraph.

If the above problem is modified in such a way that several undirected edges (streets) are prescribed to be directed, then the preceding assertion can be repeated and we obtain the following generalization of the Robbins theorem and simultaneously a slight generalization of a theorem of Boesch and Tindell [2]: *A strong multimigraph G admits introducing directions on prescribed undirected edges to produce another strong multimigraph if and only if no prescribed edge is a bridge.* On the other hand, it is sufficient to consider only the case where all undirected edges have to be directed. Indeed, if every unprescribed undirected edge is replaced by two multiple edges, then it is sufficient to direct the obtained multimigraph fully. Namely, we can suppose that no all multiple edges between two vertices will be directed in the same direction.

Sedláček [12] has proved the following strenghtened variation of the Robbins theorem: *A 2-connected multigraph G with a prescribed edge e has a strong orientation D such that $D-e$ is an acyclic multidigraph* (i.e. every cycle of D includes e). We pose the problem to find a necessary and sufficient condition for the existence of such an orientation of a 2-connected multimigraph.

Another generalization of the Robbins theorem is due to Nash-Williams [7] and concerns local line-connectivities $\lambda(u, v)$ (this value is equal to the maximum number of pairwise line-disjoint directed paths from u to v): *A multigraph G has an orientation H such that for any two vertices u and v we have* $\lambda_H(u, v) \geq [\lambda_G(u, v)/2]$. Another proof of this result can be found in [6]. Here we conjecture that the same assertion holds also for multimigraphs.

In the following two sections we deal only with diameters or radii of orientations.

2. Diameters of orientations of some special graphs

Given a graph (or multimigraph) G, $d(G)$ denotes its diameter and G^* denotes a strong orientation of G with the least possible diameter.

One of the quantitative variations of the Robbins theorem is the following result.

Lemma 2.1 (Chvátal and Thomassen [3]). If G is a bridgeless graph of diameter 2, then $d(G^*) \leqslant 6$.

Chvátal and Thomassen have also proved that the Petersen graph G has the following property:

(P) G is a bridgeless graph of diameter 2 and its every orientation has diameter at least 6.

They had known no other graph with property (P).

Theorem 2.1. The least number of vertices of a graph G with property (P) is 7.

Proof. One can easily verify that the graph of *Fig. 1* has diameter 2 and that any strong orientation of it has diameter 6. Since any digraph of diameter 6 must have at least 7 vertices, the proof follows. ■

Thus we know two graphs with property (P), but it is an open problem to find further such graphs. A long case analysis shows that there is no other such graph with 7 vertices.

If we insert $d-2$ new vertices into every edge at the top vertex of the graph of *Fig. 1*, then we obtain a bridgeless graph of diameter d such that any strong orientation of it will have diameter $3d$. However, the lower bound $d^2/2+d$ derived in [3] is better whenever $d \geqslant 5$ (see also Section 3).

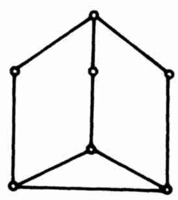

Fig. 1

Boesch and Tindell [2] have directed complete graphs and proved that $d(K_n^*)=2$ whenever $n \geqslant 3$, $n \neq 4$, and that $d(K_4^*)=3$. To see this, it is sufficient to prove that for every $n \geqslant 3$, $n \neq 4$, there exists a tournament of order n and diameter 2. As there are such tournaments for $n=3$ and 6 (see e.g. [8]) and there is a simple construction giving such a tournament with $p+2$ vertices whenever one with p vertices is known (see e.g. [8], [2]), the proof follows.

Boesch and Tindell [2] have also directed complete bipartite graphs and showed that $d(K^*_{p,p})=3$ for $p\geq 2$. Here we slightly extend their result.

Theorem 2.2. For any complete k-partite ($k\geq 2$) graph $K(n_1, n_2, \ldots, n_k)$ with $n_1\geq n_2\geq \ldots \geq n_k\geq 1$ we have

(a) $\quad d(K^*(n_1, n_2))\leq 4$ if $n_2\geq 2$,

(b) $\quad d(K^*(n_1, n_2, \ldots, n_k))\leq 3$ if either $k=2$ and $n_1=n_2\geq 2$ or $k\geq 3$,

(c) $\quad d(K^*(n_1, n_2 \ldots, n_k)=2$ if $k\geq 3$ and $n_1=n_2=\ldots=n_k\geq 2$.

Proof. One can easily verify that any strong orientation of $K(n_1, 2)$ has diameter 4 if $n_1\geq 3$. If $n_1\geq 4$ and $n_2\geq 3$ we can construct an orientation H of $K(n_1, n_2)$ from an orientation G of $K(n_1-1, n_2-1)$ as follows. Let V_1 and V_2 be the two parts of the vertex set of G. If $V(H)=V(G)\cup\{u, v\}$ then we put $E(H)=E(G)\cup\{w_1v, uw_2, vu \mid w_1\in V_1, w_2\in V_2\}$. One can easily see that $d(H)\leq d(G)$. The part (a) is proved. If we start from the strong orientation G of K (2, 2) whose diameter is 3, the first part of (b) follows too.

Let $k\geq 3$. As mentioned above, there exists a tournament T_k with k vertices and diameter at most 3. Let $V(T_k)=\{v_1, v_2, \ldots, v_k\}$ and the parts of the vertex set of $K(n_1, n_2, \ldots, n_k)$ be $V_1, V_2, \ldots, V_k$. Now we give a proper orientation as follows. Every edge xy with $x\in V_i$ and $y\in V_j$ is directed from x to y iff $v_iv_j\in E(T_k)$. The obtained k-partite tournament has the same distances between two vertices from distinct parts as the corresponding vertices in T_k have. On the other hand, the distance between two vertices from the same part V_i is 3 because there is a 3-cycle in T_k containing v_i.

To prove (c) we distinguish two cases.

Case 1. $n_1=\ldots=n_k=2$. Consider a digraph G with the property that $V(G)$ can be decomposed into two sets W_1 and W_2 such that for every $x\in W_1$ there is an arc xy with $y\in W_2$, for every $y\in W_2$ there is an arc yx with $x\in W_1$ and the distance between two vertices of the same part is at most 2. If we add to G two new vertices u and v and the arcs uw_1, w_1v, w_2u and vv_2 for all $w_1\in V_1$ and $w_2\in V_2$, we clearly obtain a new digraph H with $d(H)=2$. Letting G to be the 4-cycle K^* (2, 2), H becomes an orientation of K (2, 2, 2) with diameter 2. Generally if we have constructed an orientation $G=K^*(2, 2, \ldots, 2)$ (of the complete $(k-1)$-partite graph) with 2-vertex parts V_1; $V_2, \ldots, V_{k-1}$, we put $W_1=V_1$ and $W_2=V_2\cup\ldots\cup V_{k-1}$ and receive a k-partite digraph $K^*(2, 2, \ldots, 2)$ of diameter 2.

Case 2. $n_1=n_2=\ldots=n_k=s\geq 3$. Let $V_1, V_2, \ldots, V_k$ be the k parts (i.e., maximal independent subsets of vertices) of $G=K(n_1, \ldots, n_k)$. Denote the vertices of V_i by $v_{i1}, v_{i2}, \ldots, v_{is}$ for $i=1, 2, \ldots, k$.

Note that for every $k\geq 3$, $k\neq 4$ there is a k-vertex tournament T_k such that its every arc lies in a 3-cycle. (Namely, this is true for $k=3$ and 6 and a simple construction ensures such a $(p+2)$-vertex tournament whenever such one with p

vertices is known; see e.g. [8].) Let $V(T_k)=\{u_1, u_1, \ldots, u_k\}$. Using T_k, we now direct the edges of G as follows. For every two vertices u_i and u_j such that u_iu_j is an arc of T_k we direct the edges $v_{i1}v_{j1}$, $v_{i2}v_{j2}$, $\ldots$, $v_{is}v_{js}$ in the direction from V_i to V_j and all other edges between V_i and V_j are directed oppositely. The simple verification that the resulting orientation of G has diameter 2 is left to the reader.

It remains to consider the case $k=4$. In this case there is a unique strong tournament T_4. One sees that there is a unique arc, say u_1u_2, which does not belong to any 3-cycle but every other arc lies in a 3-cycle. Now we can direct the edges of G as above with the exception when $i=1$ and $j=2$, in which case the edges $v_{11}v_{22}$, $v_{12}v_{23}$, $\ldots$, $v_{1,s-1}v_{2s}$, $v_{1s}v_{21}$ are directed from V_1 to V_2 and all other edges between V_1 and V_2 are directed oppositely. And again, it is a routine matter to verify that the obtained digraph is of diameter 2. This completes the proof. ■

It is an open problem to determine $d(K^*(n_1, n_2, \ldots, n_k))$ in general, even the simple case when $k=2$ remains unsolved. Note, however, that $d(K^*(n_1, n_2))=4$ whenever n_1 is sufficiently large relative to n_2 (because if two vertices from the first part have equal out- or in-neighborhoods, then their distance is at least 4). For example, $K(4, 3)$ has no orientation of diameter 3, but $d(K^*(5, 4))=3$ and consequently $d(K^*(q+1, q))=3$ for any $q \geqslant 4$.

The final result of this section deals with cubes.

Theorem 2.3. Let $k \geqslant 2$ be an integer. Then for the k-dimensional cube Q_k we have $d(Q_k^*) \leqslant 2k-1$.

Proof. Noting that the assertion is trivial for $k=2$, we proceed by induction on k. The cube Q_k $(k \geqslant 3)$ consists of two copies H_1 and H_2 of Q_{k-1} and a set M of 2^{k-1} edges each of which joins a vertex v_1 of H_1 and a corresponding vertex v_2 of H_2. To direct the edges of Q_k, we simply change H_1 to H_1^* and H_2 to H_2^* (the corresponding copy of H_1^*) and the edges of M are directed as follows. Choose a vertex (as a root) u_1 of H_1 and determine distance sets $S_1(j)=\{v \mid d_{H_1}(u, v)=j\}$ for $0 \leqslant j \leqslant 2(k-1)-1$. The corresponding sets in H_2^* are denoted by $S_2(j)$. Every edge v_1v_2 between H_1 and H_2 is directed from v_1 to v_2 iff $v_1 \in S_1(j)$ where j is odd. The digraph obtained is denoted by H.

To prove that $d(H) \leqslant 2k-1$ we consider two arbitrary vertices x, y and verify that $d_H(x, y) \leqslant 2k-1$. If both x and y belong to the same H_i $(i=1, 2)$ then the induction hypothesis applies. Therefore let $x=x_1 \in H_1^*$ and $y=y_2 \in H_2^*$. If $x_1 \in S_1(j)$ with odd j, then there is an arc x_1x_2 in H and an x_2-y_2 directed path in H_2^* of length at most $2(k-1)-1$, where x_2 belongs to H_2^* and corresponds to x_1. If $x_1 \in S_1(j)$ with j even, then there are an arc x_1v_1 in H_1^*, the arc v_1v_2 (from H_1^* to H_2^*) and a v_2-y_2 path of length at most $2(k-1)-1$ in H_2^*. (Note that $v_1 \in S_1(p)$ with p odd, because Q_k is a bipartite graph.) The other cases are symmetric. Thus in each case there is a directed $x-y$ path in H of length at most $2k-1$. ■

As the diameter of Q_k is k, $k \leqslant d(Q_k^*) \leqslant 2k-1$. However, it is an open problem to determine the exact value of $d(Q_k^*)$ for each k.

Note added in proof: The exact values of $d(K^*(n_1, n_2))$ and $d(Q_k^*)$ have now been found for all parameters by my student Ľ. Šoltés.

3. Diameters and radii of orientations of weighted multigraphs

Chvátal and Thomassen [3] have established certain bounds on the diameters or radii of orientations of multigraphs. In this section we generalize some of their results to multigraphs where every edge has a length (positive weight).

Theorem 3.1. Let G be a bridgeless multigraph where every edge length is between 1 and L. If the radius of G is r, then G admits an orientation of radius at most $r^2 + rL$.

Proof. We shall follow a proof of Chvátal and Thomassen [3] and proceed by induction on r. If $r = 0$, it is nothing to prove and therefore assume that $r > 0$. Let u be a central vertex of G (i.e. $d_G(u, v) \leq r$ for every v). For any $v \in V_G(u)$ (v is a neighbour of u) define $k(v)$ to denote the length of a shortest cycle containing a shortest edge uv. Clearly,

$$k(v) \leq 2r + L . \qquad (1)$$

A multidigraph A which is an orientation of some submultigraph of G is called admissible if there is a set S of neighbors of u together with a directed cycle $C(v)$ for each $v \in S$ such that

(i) each $C(v)$ has length $k(v)$ and contains either an arc uv or an arc vu or both,
(ii) A is the union of all these cycles $C(v)$ ($v \in S$).

Now we are going to prove that

Every maximal admissible multidigraph contains all the neighbors of u. (2)

Assume the contrary, so that there exists a maximal admissible multidigraph A and a vertex $w \in V_G(u) - V(A)$. Consider a $k(w)$-cycle of G: $w_0w_1 \ldots w_t$, where $w_0 = u = w_t$ and $w_1 = w$. Let i be the smallest subscript such that $2 \leq i$ and w_i belongs to A. Thus there exists a vertex $v \in S$ with $w_i \in C(v)$ and we can assume that $C(v)$ contains an arc uv. The cycle $C(v)$ consists of two directed paths: $u - w_i$ path P_1 and $w_i - u$ path P_2. We distinguish two cases.

Case 1: The length of P_1 is less than or equal to the length of P_2. Then we add to A the directed path $P = w_iw_{i-1} \ldots w_1w_0$. The required directed cycle $C(w)$ consists of the paths P_1 and P. We assert that the length of $C(w)$ is $k(w)$. Otherwise the length is greater than $k(w)$ and then the path $P' = w_iw_{i+1} \ldots w_t$ is shorter than P_1. In that case, the submultigraph of G consisting of P_1, P and P' would produce a cycle containing uv of length less than $k(v)$, a contradiction. Adding $C(w)$ to A, we obtain a larger admissible multidigraph which contradicts the maximality of A.

Case 2: P_1 is longer than P_2. Then we add to A the directed path $P = w_0 w_1 \dots w_i$ and the required cycle $C(w)$ will consist of the paths P and P_2. Adding $C(w)$ to A, we obtain a contradiction again.

Hence (2) is proved. Consider a maximal admissible digraph A. By (1) we see that

$$\begin{aligned} d_A(u, w) &\leq 2r + L - 1\,, \\ d_A(w, u) &\leq 2r + L - 1 \end{aligned} \tag{3}$$

for every vertex w of A.

In the multigraph G, contract all the vertices of A into a new vertex $\hat{u}$ and call the resulting multigraph $\hat{G}$. Clearly, $\hat{G}$ is bridgeless and

$$d_{\hat{G}}(\hat{u}, v) \leq r - 1 \tag{4}$$

for every vertex v of $\hat{G}$.

Thus the radius of $\hat{G}$ is at most $r - 1$ and by the induction hypothesis there is an orientation $\hat{H}$ of $\hat{G}$ such that

$$\begin{aligned} d_{\hat{H}}(\hat{u}, v) &\leq (r-1)^2 + (r-1)L\,, \\ d_{\hat{H}}(v, \hat{u}) &\leq (r-1)^2 + (r-1)L \end{aligned} \tag{5}$$

for every vertex v of $\hat{H}$.

If we expand the vertex $\hat{u}$ to $V(A)$, then the edges of A together with those corresponding to the edges of $\hat{H}$ will form an orientation H_0 of a submultigraph of G. Directing all the remaining edges of G arbitrarily, we obtain an orientation H of G. By (3) and (5) we have

$$d_H(u, v) \leq (r-1)^2 + (r-1)L + 2r + L - 1 = r^2 + rL$$

and analogously

$$d_H(v, u) \leq r^2 + rL$$

for every vertex v of H.

Thus the radius of H does not exceed $r^2 + rL$. ∎

It is well known that $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$ and thus we have the following corollary of Theorem 3.1.

Theorem 3.2. Let G be a bridgeless multigraph with all edge lengths between 1 and L. If the diameter of G is d, then G admits an orientation of diameter at most $2d^2 + 2dL$.

We conjecture that Theorems 3.1. and 3.2 remain valid also for multimigraphs.

The following result shows that Theorem 3.1 is in some sense the best possible but as for Theorem 3.2, there is a gap which remains unsolved (see Theorem 3.4 below).

Theorem 3.3. For every positive integer r and any real number $L \geqslant 1$ there is a bridgeless graph G_r with every edge length equal to 1 or L and such that G_r has radius r and every orientation of G_r has radius at least $r^2 + rL$.

Proof. At first we construct a certain sequence $H_1, H_2, \ldots$ of rooted graphs. H_1 is a triangle with one of its vertices designed as the root u_1; the opposite edge has length L and the remaining 2 edges are of length 1. If $H_1, H_2, \ldots, H_{r-1}$ $(r \geqslant 2)$ are known, then H_r is constructed as follows. Take a cycle with $2r+1$ vertices, designate one of its vertices as the root u_r and assign edge length L to the opposite edge and length 1 to every other edge. Let u, v be the two neighbors of the root u_r. Amalgamate the cycle with two copies of H_{r-1} by identifying the root of the first copy with u and the root of the second copy with v. The resulting graph, rooted at u_r, is H_r.

Now, G_r is obtained by taking two copies of H_r identifying their roots. *Fig. 2* shows the graph G_3. One can easily verify that rad $(G_r) = r$ (u_r is the unique central

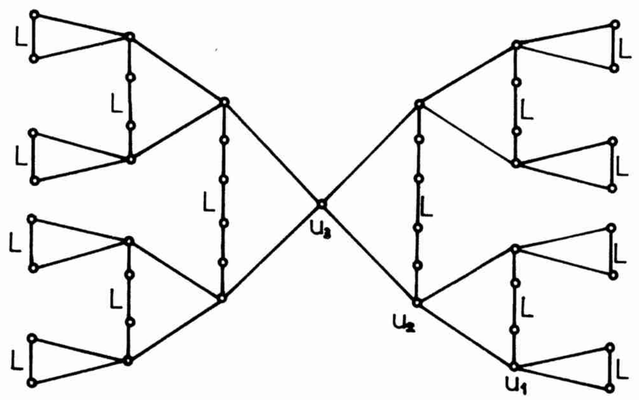

Fig. 2

vertex). In any strong orientation (all are isomorphic) of G_r each cycle of G_r becomes a directed cycle. Thus the radius will be equal to

$$(2r-1+L)+(2r-3+L)+\ldots+(3+L)+(1+L)=r^2+rL. \blacksquare$$

Theorem 3.4. For every integer $d \geqslant 2$ and any real number $L \geqslant 1$ there is a bridgeless graph G^d with every edge length equal to 1 or L and such that the diameter of G^d is d and every strong orientation of G^d has diameter $d^2/2 + dL$ if d is even and diameter $d^2/2 + dL + 3/2 - d$ if d is odd.

Proof. If d is even, then put $r = d/2$ and consider the graph G_r from the proof of Theorem 3.3. We see that $\operatorname{diam}(G_r) = 2r = d$ and any strong orientation of G_r has diameter $2(r^2 + rL) = d^2/2 + dL$. Thus we can take $G^d = G_r$. Now let d be odd. Then $d = 2r+1$ with $r \geqslant 1$ and we construct G^d as follows. Consider the graph G_r

again and denote by x one of the 2^{r+1} peripheral vertices lying in triangles. Add to G_r two new vertices y and z and three new edges: xy and xz with lengths equal to 1, and yz with length L. This gives a graph G^d with diameter $d=2r+1$. Clearly, every strong orientation of G^d has diameter $2(r^2+rL)+1+L = d^2/2+dL+ +3/2-d$. ■

If highly connected graphs are required, we can give at least the following result.

***Theorem* 3.5.** (*a*) *For all integers* $r \geqslant 1$, $k \geqslant 1$ and any real number $L \geqslant 1$ there is a k-vertex-connected graph $G_{r,k}$ with every edge length equal to 1 or L and such that the radius of $G_{r,k}$ is r and its every orientation has radius at least $r^2/2+r(L+1)/2$.

(b) For all integers $d \geqslant 2$, $k \geqslant 1$ and any real number $L \geqslant 1$ there is a k-vertex-connected graph $G^{d,k}$ with every edge length equal to 1 or L and such that the diameter of $G^{d,k}$ is d and its every orientation has diameter at least $d^2/4+d(L+1)/2$.

Proof. We shall use "a branch" $\hat{H}_r$ of the graph H_r constructed in the proof of Theorem 3.3. $\hat{H}_r$ consists of r cycles $C_{2r+1}, C_{2r-1}, \ldots, C_3$ properly amalgamated as shown in *Fig. 3*, where $\hat{H}_3$ is depicted. Clearly, the distance $d(u_r, v) \leqslant r$ for each

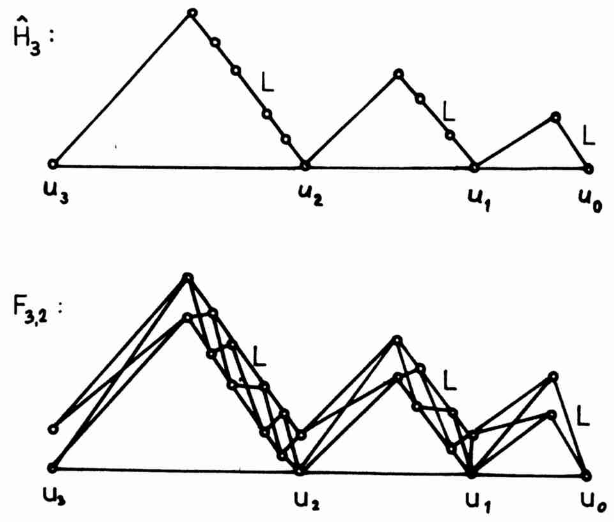

Fig. 3

vertex v of $\hat{H}_r$. Now to every vertex x of $\hat{H}_r$, $x \neq u_0$, $k-1$ new vertices are added such that x is substituted by a set $M(x)$ of k vertices; we put $M(u_0)=\{u_0\}$. Every edge $xy \neq u_iu_{i+1}$ $(i=0, 1, \ldots, r-1)$ of $\hat{H}_r$ is substituted by $|M(x)|\ |M(y)|$ edges of the same length, i.e. each $v \in M(x)$ is joined to each $w \in M(y)$. Moreover, we join u_1 to every other vertex of $M(u_1)$. Between $M(u_i)$ and $M(u_{i+1})$ it remains the unique edge u_iu_{i+1}. Call the resulting graph $F_{r,k}$ (for $F_{3,2}$ see *Fig. 3*).

Now we are prepared to prove the desired assertions. First we give the proof for radius and for even diameter. Take two copies $F'_{r,k}$ and $F''_{r,k}$ of $F_{r,k}$. Identifying their sets $M(u'_r)$ and $M(u''_r)$, we obtain a graph $G_{r,k}$, which is obviously k-vertex-connected. The graph $G_{3,2}$ is roughly depicted in *Fig. 4*. One sees that $G_{r,k}$ has radius r (with the central vertex $u'_r = u''_r$) and diameter $d = 2r$ ($d(u'_0, u''_0) = 2r$).

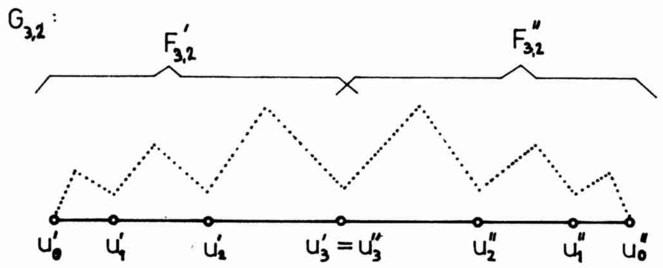

Fig. 4

For every strong orientation of $F_{r,k}$ and for any two shortest directed paths P_1 going from u_0 to a vertex of $M(u_r)$ and P_2 going from a vertex of $M(u_r)$ to u_0 we have: If P_1 contains the arc $u_{i-1}u_i$, then the section of P_2 between $M(u_i)$ and $M(u_{i-1})$ has length at least $2i + L - 1$ and conversely. Thus the sum of lengths of P_1 and P_2 is at least

$$\sum_{i=1}^{r} [1 + (2i + L - 1)] = r^2 + r + rL .$$

Therefore for any strong orientation H of $G_{r,k}$ we have: $d_H(u'_0, u''_0) + d_H(u''_0, u'_0) \geqslant 2r^2 + 2r(L+1)$. Hence the diameter of H is at least $r^2 + r(L+1) = d^2/4 + d(L+1)/2$ and we can put $G^{2r,k} = G_{r,k}$. Consequently, the radius of H is at least $r^2/2 + r(L+1)/2$.

Now, let d be odd, i.e. there is a positive integer r with $d = 2r+1$. Take a copy $F'_{r,k}$ of $F_{r,k}$ and a copy $F''_{r+1,k}$ of $F_{r+1,k}$. Amalgamating $F'_{r,k}$ and $F''_{r+1,k}$ by identifying $M(u'_r)$ and $M(u''_{r+1})$, we obtain a k-vertex-connected graph $G^{2r+1,k}$ of diameter $2r+1$. Reasoning as above, one can easily verify that any strong orientation of $G^{2r+1,k}$ will have diameter at least

$$[r^2 + r(L+1) + (r+1)^2 + (r+1)(L+1)]/2 = \\ = \left[\frac{d^2}{2} + \frac{1}{2} + d(L+1)\right]/2 > d^2/4 + d(L+1)/2. \ \blacksquare$$

BIBLIOGRAPHY

[1] Behzad, M.—Chartrand, G.—Lesniak-Foster, L.: Graphs and digraphs, Prindle, Boston 1979.

[2] Boesch, F.—Tindell, R.: Robbins's theorem for mixed multigraphs, Amer. Math. Monthly 87 (1980), 716—719.

[3] Chvátal, V.—Thomassen, C.: Distances in orientations of graphs, J. Combinatorial Theory B 24 (1978), 61—75.

[4] Harary, F.: Graph theory, Addison-Wesley, Reading 1969.

[5] Lambin, N. V.—Tanaev, V. S.: The contour-free orientation of composite graphs (Russian), Dokl. Akad. Nauk BSSR 14 (1970), 780—781.

[6] Mader, W.: Eine Reduktionsmethode für den Kantenzusammenhang in Graphen, Math. Nachr. 93 (1979), 187—204.

[7] Nash-Williams, C. St. J. A.: On orientations, connectivity and odd-vertex-pairings in finite graphs, Canad. J. Math. 12 (1960), 555—567.

[8] Plesník, J.: Diametrically critical tournaments, Časop. Pěst. Mat. 100 (1975), 361—370.

[9] Plesník, J.: Grafové algoritmy, Veda, Bratislava 1983.

[10] Robbins, H. E.: A theorem on graphs with an application to a problem of traffic control, Amer. Math. Monthly 46 (1939), 281—283.

[11] Roberts, F. S.: Discrete mathematical models, Prentice-Hall, Englewood Cliffs, N. J., 1976.

[12] Sedláček, J.: O jednom typu dobře orientovaných grafů, Časop. Pěst. Mat. 84 (1959), 7—15.

Received: 12. 9. 1983

Author's address:
Ján Plesník
Katedra numerických a optimalizačných metód MFF UK
Mlynská dolina
842 15 Bratislava

SÚHRN

POZNÁMKY O DIAMETROCH ORIENTÁCIÍ GRAFOV

Ján Plesník, Bratislava

Rozširujú sa niektoré výsledky z prác [2] a [3]. Uvádza sa najmenší graf bez mostov a s diametrom 2, ktorého každá orientácia má diameter aspoň 6. Dané sú horné ohraničenia na minimálny diameter orientácie kompletného k-partitného grafu a k-rozmernej kocky. Tiež sa skúmajú ohodnotené multigrafy daného diametra alebo rádiusa a trochu sa zovšeobecňujú niektoré výsledky z [3]. V článku sa uvádzajú viaceré otvorené problémy a hypotézy.

РЕЗЮМЕ

ЗАМЕТКИ О ДИАМЕТРАХ ОРИЕНТАЦИЙ ГРАФОВ

Ян Плесник, Братислава

Расширяются некоторые результаты работ [2] и [3]. Предлагается наименьший граф без мостов и с диаметром 2, каждая ориентация которого имеет диаметр по меньшей мере 6. Даны верхние оценки для минимального диаметра ориентации полного k-дольного графа и k-куба. Тоже исследуются взвешенные мультиграфы с данным диаметром или радиусом и в некоторой степени обобщаются результаты из [3]. В работе приводится несколько нерешенных задач и предложений.