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INTEGRAL EQUIVALENCE OF MULTIVALUED DIFFERENTIAL SYSTEMS

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The following systems are considered

$$x' = f(t, x) \tag{a}$$

and

$$y' \in F(t, y), \tag{b}$$

where f is such that it guarantees the existence of solutions of (a) on the infinite interval $\langle 0, \infty)$; $F(t, y)$ is a nonempty compact convex subset of E_n for each (t, y) from $\langle 0, \infty) \times E_n$ and F also guarantees the existence of solutions of (b) on the infinite interval $\langle 0, \infty)$.

By a solution of (b), we mean an absolutely continuous function on some nondegenerate subinterval of $\langle 0, \infty)$ which satisfies (b) almost everywhere (a.e.).

Definition 1. Let $\psi(t)$ be a positive continuous function on an interval $\langle t_0, \infty)$ and let $p>0$. We shall say that two systems (a) and (b) are (ψ, p)-integral equivalent on $\langle t_0, \infty)$ iff for each solution $x(t)$ of (a) there exists a solution $y(t)$ of (b) such that

$$\psi^{-1}(t)\,|x(t)-y(t)| \in L_p(t_0, \infty) \tag{c}$$

and conversely, for each solution $y(t)$ of (b) there exists a solution $x(t)$ of (a) such that (c) holds.

By a restricted (ψ, p)-integral equivalence between (a) and (b) we shall mean that the relation (c) is satisfied for some subsets of solutions of (a) and (b), e.g. for the bounded solutions.

We shall say that a function $z(t)$ is ψ-bounded on the interval $\langle t_0, \infty)$ iff

$$\sup_{t \geq t_0} |\psi^{-1}(t) z(t)| < \infty .$$

Next we shall consider special systems

$$x' \in A(t)x + F(t, x) \tag{1}$$

and

$$y' = A(t)y\ , \tag{2}$$

where $A(t)$ is an $n \times n$ matrix-function defined on $\langle 0, \infty)$ whose elements are integrable on compact subsets of $\langle 0, \infty)$; x and y are n-dimensional vectors, and $F(t, x)$ is a nonempty compact and convex subset of E_n for each $(t, x) \in \langle 0, \infty) \times E_n$.

Analogues of our result in ordinary differential equations may be found in [2], [4] and [7].

Now we shall fix notations, introduce notions and state lemmas which will be needed in the sequel:

We shall write $|.|$ for any convenient matrix (vector) norm in E_n. If A is a subset of E_n, we shall denote

$$|A| = \sup \{|a| : a \in A\}\ .$$

$L_p^n(J)$ will denote the n-th Cartesian product of $L_p(J)$. $B(t_0)$ will denote the space of continuous functions from $\langle t_0, \infty)$ to E_n. Let $\psi(t)$ be a positive continuous function on $\langle t_0, \infty)$. For $z \in B(t_0)$, we denote $|z|_\psi = \sup_{t \geq t_0} |\psi_{(t)}^{-1} z(t)|$. Let $B_\psi = \{z \in B(t_0) : |z|_\psi < \infty\}$. Then B_ψ with norm $|.|_\psi$ is a Banach space. For $\varrho > 0$, we denote $B_{\psi, \varrho} = \{z \in B(t_0) : |z|_\psi \leq \varrho\}$. For a topological vector space Y let cf(Y) denote the set of all closed convex nonempty subsets of Y.

Let X and Y be topological spaces.

Definition 2. A mapping F: $X \to 2^Y$ is said to be upper-semicontinuous at the point $x \in X$, if for arbitrary neighbourhood U of the set-image $F(x)$ there exists such neighbourhood V of the point x that $F(V) \subset U$, where $F(V) = \bigcup_{z \in V} F(z)$. This mapping is said to be upper-semicontinuous if it is upper-semicontinuous in each point $x \in X$.

Definition 3 (Definition 4, W. Sobieszek [5]).

We say that the mapping F: $X \to 2^Y$ is upper semicompact at the point $x \in X$ if from the assumptions $x_n \to x$, $x_n \in X$, $y_n \in F(x_n)$ it follows that there exists a subsequence of the sequence $\{y_n\}$, convergent to some $y \in F(x)$.

Definition 4 (Definition 3′, W. Sobieszek, P. Kowalski [6]). The map F is upper semicompact at a point $x \in X$ iff F is i) upper semicontinuous at the point x and ii) the set $F(x)$ is compact.

Lemma 1 (W. Sobieszek, P. Kowalski [6]). Let X fulfil the first axiom of countability and Y fulfill the second axiom of countability. Then F is upper semicompact at a point x in the sense of Definition 3 if and only if it is upper semicompact at a point x in the sense of Definition 4.

Lemma 2. Let $p \geqslant 1$ and $g(t), f(t)$ be nonnegative functions for $t \geqslant 0$. Then

$$\left(\int_0^\infty g(t)\left(\int_t^\infty f(s)\,\mathrm{d}s\right)^p \mathrm{d}t\right)^{1/p} \leqslant \left(\int_0^\infty f(s)\left(\int_0^s g(t)\,\mathrm{d}t\right)^{1/p} \mathrm{d}s\,.\right.$$

Proof. Let $\chi_{\langle 0, s\rangle}$ be the characteristic function of the interval $\langle 0, s\rangle$. Then using the Minkowski inequality we get

$$\left(\int_0^\infty g(t)\left(\int_t^\infty f(s)\,\mathrm{d}s\right)^p \mathrm{d}t\right)^{1/p} =$$

$$\left(\int_0^\infty \left(\int_0^\infty g(t)^{1/p} f(s)\chi_{\langle 0, s\rangle}(t)\,\mathrm{d}s\right)^p \mathrm{d}t\right)^{1/p} \leqq$$

$$\leqq \int_0^\infty \left(\int_0^\infty \left(g^{1/p}(t) f(s)\chi_{\langle 0, s\rangle}(t)\right)^p \mathrm{d}t\right)^{1/p} \mathrm{d}s =$$

$$= \int_0^\infty f(s)\left(\int_0^\infty (g(t)\chi_{\langle 0, s\rangle}(t)\,\mathrm{d}t\right)^{1/p} \mathrm{d}s =$$

$$= \int_0^\infty f(s)\left(\int_0^s g(t)\,\mathrm{d}t\right)^{1/p} \mathrm{d}s\,.$$

Corollary 1. Let $p \geqslant 1$ and $f(t)$ be nonnegative function for $t \geqslant 0$. Then

$$\left(\int_0^\infty \left(\int_t^\infty f(s)\,\mathrm{d}s\right)^p \mathrm{d}t\right)^{1/p} \leqslant \int_0^\infty s^{1/p} f(s)\,\mathrm{d}s\,.$$

Lemma 3. (Lemma 3, A. Haščák, M. Švec [4].)

Let $\psi(t)$ and $\varphi(t)$ be positive functions for $t \geqslant 0$, $Y(t)$ a nonsingular matrix and P a projection.

Further, suppose that

$$\left[\int_0^t |\psi^{-1}(t) Y(t) P Y^{-1}(s)\varphi(s)|^p\,\mathrm{d}s\right]^{1/p} \leqslant K$$

for $t \geqslant 0$, $K > 0$, $p > 0$ and

$$\int_0^\infty \exp\left(-\mathrm{K}^{-p}\int_0^t \varphi^p(s)\psi^{-p}(s)\,\mathrm{d}s\right)\mathrm{d}t < \infty\,.$$

Then

$$\lim \psi^{-1}(t)\,|Y(t)P| = 0 \quad \text{as} \quad t \to \infty$$

and

$$|\psi^{-1}(t) Y(t) P| \in L_p(0, \infty)\,.$$

We assume throughout this paper that $F\colon \langle 0, \infty) \times E_n \to \mathrm{cf}(E_n)$ satisfies the following conditions:

(c_1) for every fixed $t \in \langle 0, \infty)$ the function $F(t, x)$ is upper — semicontinuous,

(c_2) for each measurable function $x\colon \langle 0, \infty) \to E_n$, there exists a measurable function $f_x\colon \langle 0, \infty) \to E_n$ such that $f_x(t) \in F(t, x(t))$ a.e. on $\langle 0, \infty)$.

Given a function $x \in B_\psi$ denote by $M(x)$ the set of all measurable functions

$$y\colon \langle t_0, \infty) \to E_n \quad \text{such that}$$

$$y(t) \in F(t, x)) \quad \text{a.e. on } \langle t_0, \infty).$$

Theorem 1. Let $\psi(t)$ and $\varphi(t)$ be a positive continuous functions for $t \geq 0$.
Suppose that there exists $g\colon \langle 0, \infty) \times \langle 0, \infty) \to \langle 0, \infty)$ such that

(i) $g(t, u)$ is monotone nondecreasing in u for each fixed $t \in \langle 0, \infty)$ and integrable on compact subsets of $\langle 0, \infty)$ for fixed $u \in \langle 0, \infty)$,

(ii) $\int_0^\infty g^{p'}(s, c)\, ds < \infty$ for any constant $c \geq 0$, $p' \geq 1$

(iii) for each $x \in E_n$

$$|F(t, x)| \leq \varphi(t) g(t, \psi^{-1}(t)|x|) \quad \text{a.e. on } \langle 0, \infty).$$

Then the correspondence $x \to M(x)$ defines a bounded mapping of $B_{\psi,\varrho}$ into $\mathrm{cf}(L^n_{p',\varphi})$ where

$$L^n_{p',\varphi} = \left\{ y\colon |y|_{p',\varphi} = \left[\int_{t_0}^\infty |\varphi^{-1}(s) y(s)|^{p'}\, ds \right] < \infty \right\}.$$

Proof. We have to show that for every $x \in B_{\psi,\varrho}$: (a) $M(x)$ is not empty, (b) $M(x)$ is convex, (c) $M(x) \subset L^n_{p',\varphi}$, (d) $M(x)$ is closed, (e) for every positive constant K there is a constant N such that $|x|_\psi \leq K$ implies $|y|_{p',\varphi} \leq N$ for every $y \in M(x)$. (a) and (b) are trivial, (e) follows from the assumption (ii) and (iii) and obviously implies (c).

Thus we have to prove (d) only. Let $\{y_n\}$, $y_n \in M(x)$ be the sequence such that $|y_n - y|_{p',\varphi} \to 0$ as $n \to \infty$. By Riesz theorem there is a subsequence $\{y_{1n}\}$ of the sequence $\{y_n\}$ such that $\{y_{1n}(t)\}$ converges a.e. on $\langle t_0, \infty)$ to $y(t)$ as $n \to \infty$. Since $y_{1n}(t) \in F(t, x(t))$ a.e. on $\langle t_0, \infty)$, we have $y(t) \in F(t, x(t))$ a.e. on $\langle t_0, \infty)$ i.e. $y(t) \in M(x)$.

Definition 5. Let X be a topological space and let Y be a normed linear space. A mapping $F\colon X \to 2^Y$ is weakly upper semicompact at a point $x \in X$ iff from the assumptions $x_n \in X$, $x_n \to x$, $y_n \in F(x_n)$ it follows that there is a subsequence of the sequence $\{y_n\}$ which weakly converges to some $y \in F(x)$.

Theorem 2. Let the hypotheses of Theorem 1 be satisfied. Then the mapping $M\colon B_{\psi,\varrho} \to \mathrm{cf}(L^n_{p',\varphi})$ is weakly upper semicompact.

Proof. Let $x_n,\ x \in B_{\psi,\varrho}$, $|x_n - x|_\psi \to 0$ as $n \to \infty$ and $y_n \in M(x_n)$. Since

$$|y_n|_{p',\varphi} \leq c = \left(\int_{t_0}^{\infty} g^{p'}(s,\varrho)\, ds\right)^{1/p'},$$

there is a subsequence $\{y_{1n}\}$ of the sequence $\{y_n\}$, which weakly converges to some $y \in L^n_{p',\varphi}$. We only have to prove that $y \in M(x)$ .By Banach—Saks Theorem, there is a subsequence $\{y_{2n}\}$ of the sequence $\{y_{1n}\}$ such that

$$\left|\frac{1}{n^2}\sum_{k=1}^{n^2} y_{2k} - y\right|_{p',\varphi} \to 0 \text{ for } n \to \infty .$$

Because of

$$\left|\frac{1}{n^2}\sum_{k=1}^{n-1} y_{2k}\right|_{p',\varphi} \to 0, \quad \left|\frac{1}{n^2}\sum_{k=n^2+1}^{n^2+n-1} y_{2k}\right|_{p',\varphi} \to 0$$

as $n \to \infty$, we have that

$$\left|\frac{1}{n^2}\sum_{k=n}^{n^2+n-1} y_{2k} - y\right|_{p',\varphi} \to 0 \text{ as } n \to \infty .$$

Now, by Riesz Theorem, there is a sequence $\{\sigma_n\}$, $\sigma_n \in N$, $\sigma_n \geq n$ such that

$$\frac{1}{\sigma_n^2}\sum_{k=\sigma_n}^{\sigma_n^2+\sigma_n-1} y_{2k}(t) \to y(t) \text{ a.e. on } \langle t_0, \infty) \text{ for } n \to \infty .$$

On the other hand, by the assumption (c_1), for almost every fixed $t \in \langle t_0, \infty)$ and any $\varepsilon > 0$ there is an integer $N(\varepsilon, t)$ such that

$$F(t, x_i(t)) \subset F(t, x(t)) + K_\varepsilon = \{u + v\colon u \in F(t, x(t)),\ |v| \leqq \varepsilon\}$$

for $i \geq N(\varepsilon, t)$.

Thus $y_{2k}(t) \in F(t, x(t)) + K_\varepsilon$, $2k \geq N(\varepsilon, t)$ and by convexity of $F(t, x(t))$

$$\frac{1}{\sigma_n^2}\sum_{k=\sigma_n}^{\sigma_n^2+\sigma_n-1} y_{2k} \in F(t, x(t)) + K_\varepsilon,\quad 2\sigma_n \geq N(\varepsilon, t)$$

so that

$$y(t) \in F(t, x(t)) \text{ a.e. on } \langle t_0, \infty) .$$

Theorem 3. Let the hypotheses of Theorem 1 be satisfied. Let $Y(t)$ be a fundamental matrix of (2), let there exist supplementary projectors P_1, P_2, a constant $K > 0$ and $1 < p < \infty$ such that

(iv)
$$\left[\int_0^t |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\varphi(s)|^p\, ds + \int_t^{\infty} |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|^p\, ds\right]^{1/p} \leq K$$

for all $t \geq 0$.

Then the operators L_iM: $B_{\psi,\varrho}\to\mathrm{cf}(B)$, $i=1, 2$, defined by

$$L_1Mx(t)=\left\{z\colon z=\int_{t_0}^{t} Y(t)P_1Y^{-1}(s)f_x(s)\,\mathrm{d}s,\ x\in B_{\psi,\varrho}\ \text{an}\ f_x\in M(x)\right\}$$

and

$$L_2Mx(t)=\left\{z\colon z=\int_{t}^{\infty} Y(t)P_2Y^{-1}(s)f_x(s)\,\mathrm{d}s,\ x\in B_{\psi,\varrho}\ \text{and}\ f_x\in M(x)\right\}$$

are upper semicontinuous.

Proof. Because of Lemma 1, it suffices to prove that the operators L_iM $i=1, 2$, are upper semicompact. We shall show this for the operator L_2M only. In the same way we can prove that L_1M is also upper semicompact.

Let $|x_n-x|_\psi\to 0$, x_n, $x\in B_{\psi,\varrho}$ and $z_n\in L_2Mx_n$. We have to show that there is a subsequence of $\{z_n\}$ which converges to some $z\in L_2Mx$ (in the norm $|\,.\,|_\psi$) as $n\to\infty$.

Let $z_i=\int_t^\infty Y(t)P_2Y^{-1}(s)y_i(s)\,\mathrm{d}s$, $y_i\in M(x_i)$

(existence of $\int_t^\infty Y(t)P_2Y^{-1}(s)y_i(s)\,\mathrm{d}s$ is guaranteed by (iii) and (iv)).

Since $M(x)$ is weakly upper semicompact, there is a subsequence $\{y_{1i}\}$ of $\{y_i\}$ which weakly converges to some $y\in M(x)$, i. e.

$$z_{1i}(t)\to z(t)=\int_t^\infty Y(t)P_2Y^{-1}(s)\,y(s)\,\mathrm{d}s\in L_2Mx(t)$$

a.e. on $\langle t_0,\infty)$ as $i\to\infty$.

Further, using the Hölder inequality (iv) and (iii) we get

$$\begin{aligned}|z_{1i}(t)|\leqq&\left|\psi(t)\int_{t_0}^\infty \psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)\varphi^{-1}(s)y_{1i}(s)\,\mathrm{d}s\right|\leqslant\\ &\leqslant\psi(t)K\left[\int_{t_0}^\infty|\varphi^{-1}(s)y_{1i}(s)|^{p'}\,\mathrm{d}s\right]^{1/p'}\leqslant\\ &\leqslant\psi(t)K\left[\int_{t_0}^\infty g^{p'}(s,\varrho)\,\mathrm{d}s\right]^{1/p'},\end{aligned}$$

thus the functions z_{1i}, $i=1, 2, \ldots,$, are uniformly ψ-bounded and by virtue of

$$\begin{aligned}|z_{1i}(t_1)-z_{1i}(t_2)|=&\left|\int_{t_1}^{t_2} Y(t)P_2Y^{-1}(s)y_{1i}(s)\,\mathrm{d}s\right|\leqslant\\ \leqslant\psi(t)&\left[\int_{t_1}^{t_2}|\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|^p\,\mathrm{d}s\right]^{1/p}\cdot\left[\int_{t_1}^{t_2}g^{p'}(s,\varrho)\,\mathrm{d}s\right]^{1/p'}\leqslant\\ &\leqslant\psi(t)K\left[\int_{t_1}^{t_2}g^{p'}(s,\varrho)\,\mathrm{d}s\right]^{1/p'},\quad t_1\leqslant t_2,\end{aligned}$$

they are also equicontinuous on every compact subinterval of $\langle t_0, \infty)$. By Ascoli Theorem, as well as Cantor's diagonalization process, the sequence $\{z_{1i}\}$ contains a subsequence $\{z_{2i}\}$, which is uniformly convergent on every compact subinterval of $\langle t_0, \infty)$. This fact together with the inequality

$$\psi^{-1}(t)|z_{2i}(t)| \leqslant K\left[\int_t^\infty g^{p'}(s, \varrho)\, ds\right]^{1/p'}$$

guarantees the convergence of $\{z_{2i}\}$ on $\langle t_0, \infty)$ in the norm $|.|_\psi$.

Now we are able to prove some theorems concerning the (ψ, p) — integral equivalence of the systems (1), (2).

Theorem 4. Let $Y(t)$ be a fundamental matrix of (2), $\psi(t)$ and $\varphi(t)$ are positive continuous functions for $t \geqslant 0$.

Suppose that

a) there exist supplementary projections P_1, P_2 and constant $K>0$ and $2 \leqslant p < \infty$ such that

$$\left[\int_0^t |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\varphi(s)|^p\, ds + \right.$$
$$\left. + \int_t^\infty |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|^p\, ds\right]^{1/p} \leqslant K$$

for all $t \geqslant 0$,

b) there exists g: $\langle 0, \infty) \times \langle 0, \infty) \to \langle 0, \infty)$ such that (i) $g(t, u)$ is monotone nondecreasing in u for each fixed $t \in \langle 0, \infty)$ and integrable on compact subsets of $\langle 0, \infty)$ for fixed $u \in \langle 0, \infty)$, (ii) $\int_0^\infty s^{1/p} g^{p'}(s, c)\, ds < \infty$ for any constant $c \geqslant 0$, where $1/p + 1/p' = 1$, (iii) for each $x \in E_n$

$$|F(t, x)| \leqslant \varphi(t) g(t, \psi^{-1}(t)|x|) \text{ a.e. on } \langle 0, \infty),$$

c) $$\int_0^\infty \exp\left\{-K^{-p}\int_0^t \varphi^p(s)\psi^{-p}(s)\, ds\right\} dt < \infty,$$

d) $$\int_0^\infty |P_1Y^{-1}(s)\varphi(s)| g(s, c)\, ds < \infty.$$

Then the set of ψ-bounded solutions of (1) and (2) are (ψ, p)-integral equivalent.

Proof. Let $y(t)$ be a ψ-bounded solution of (2) on $\langle t_0, \infty)$, $t_0 \geqslant 0$. Then there is $\varrho > 0$ such that $y \in B_{\psi, \varrho}$.

Define for $x \in B_{\psi, 2\varrho}$ the operator

$$Tx(t) = \left\{z \in B_{\psi, 2\varrho}\colon z(t) = y(t) + \int_{t_0}^t Y(t)P_1Y^{-1}(s)f_x(s)\, ds - \right.$$
$$\left. - \int_t^\infty Y(t)P_2Y^{-1}(s)f_x(s)\, ds,\ f_x \in M(x)\right\}.$$

The existence of

$$\int_t^\infty Y(t)P_2Y^{-1}(s)f_x(s)\,\mathrm{d}s$$

is guaranteed by a) and b).

For each $x \in B_{\psi,2\varrho}$ is $M(x)$ equipped with the norm $|\,.\,|_{p',\varphi}$ defined by

$$|f_x|_{p',\varphi} = \left[\int_{t_0}^\infty |\varphi^{-1}(s)f_x(s)|^{p'}\,\mathrm{d}s\right]^{1/p'}.$$

By Theorem 2 $LM=(L_1-L_2)M$ maps $B_{\psi,2\varrho}$ into $\mathrm{cf}(B_\psi)$ and is upper semicontinuous on $B_{\psi,2\varrho}$. Because $y \in B_\psi$ is fixed, we have that also $Tx \in \mathrm{cf}(B_\psi)$ and T is upper semicontinuous on $B_{\psi,2\varrho}$.

Further for each $z \in Tx$, $x \in B_{\psi,2\varrho}$, it is valid

$$\begin{aligned}
&|\psi^{-1}(t)z(t)| \le \\
&\le |\psi^{-1}(t)y(t) + \int_{t_0}^t |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\varphi(s)|\,g(s,2\varrho)\,\mathrm{d}s + \\
&\quad + \int_t^\infty |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|\,g(s,2\varrho)\,\mathrm{d}s \le \\
&\le \varrho + \left[\int_{t_0}^t |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\varphi(s)|^p\,\mathrm{d}s\right]^{1/p}\cdot\left[\int_{t_0}^t g^{p'}(s,2\varrho)\,\mathrm{d}s\right]^{1/p'} + \\
&\quad + \left[\int_t^\infty |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|^p\,\mathrm{d}s\right]^{1/p}\cdot\left[\int_t^\infty g^{p'}(s,2\varrho)\,\mathrm{d}s\right]^{1/p'} \le \\
&\le \varrho + K\left[\int_{t_0}^\infty g^{p'}(s,2\varrho)\,\mathrm{d}s\right]^{1/p'}.
\end{aligned}$$

If we choose t_0 such that

$$K\left\{\int_{t_0}^\infty g^{p'}(s,2\varrho)\,\mathrm{d}s\right\}^{1/p'} \le \varrho\,,$$

we have that T maps $B_{\psi,2\varrho}$ into itself.

Let $x \in B_{\psi,2\varrho}$ and $z \in T(x)$. Then there is $f_x \in M(x)$ such that

$$z' = A(t)z + f_x(t) \quad \text{a.e. on } \langle t_0, \infty)\,.$$

Therefore by b) (iii) for $t_0 \le t_1 \le t_2$

$$\begin{aligned}
|z(t_1)-z(t_2)| &\le \int_{t_1}^{t_2} |A(s)|\,|z(s)|\,\mathrm{d}s + \int_{t_1}^{t_2} |f_x(s)|\,\mathrm{d}s \le \\
&\le 2\varrho\int_{t_1}^{t_2} |A(s)|\psi(s)\,\mathrm{d}s + \int_{t_1}^{t_2} \varphi(s)g(s,2\varrho)\,\mathrm{d}s\,.
\end{aligned}$$

Thus the functions in $TB_{\psi,2\varrho}$ are equicontinuous on every compact subinterval of $\langle t_0, \infty)$.

Then K. Fan's fixed point theorem (K. Fan [1]) yields the existence of $x \in B_{\psi, 2\varrho}$ such that $x \in T(x)$. Clearly this fixed point $x(t)$ is a ψ-bounded solution of (1).

Conversely, let $x(t)$ be a ψ-bounded solution of (1). Define for $t \geq t_0 \geq 0$

$$y(t) = x(t) - \int_{t_0}^{t} Y(t)P_1 Y^{-1}(s) f_x(s)\, \mathrm{d}s + \int_{t}^{\infty} Y(t)P_2 Y^{-1}(s) f_x(s)\, \mathrm{d}s ,$$

where

$$f_x(t) = x'(t) - A(t)x(t) \in F(t, x(t)) \text{ a.e. on } \langle t_0, \infty) .$$

It is easy to prove that $y(t)$ is a ψ-bounded solution of (2). Now we have to proves that

$$\psi^{-1}(t)\,|x(t) - y(t)| \in L_p(t_0, \infty) .$$

We have

$$\psi^{-1}(t)[x(t) - y(t)] =$$
$$= \int_{t_0}^{t} \psi^{-1}(t) Y(t) P_1 Y^{-1}(s) f_x(s)\, \mathrm{d}s - \int_{t}^{\infty} \psi^{-1}(t) Y(t) P_2 Y^{-1}(s) f_x(s)\, \mathrm{d}s .$$

It is sufficient to show that the terms on the right-hand side belong to $L_p(t_0, \infty)$.

By the assumptions of the theorem and the Hölder inequality we get

$$\left| \int_{t_0}^{t} \psi^{-1}(t) Y(t) P_1 Y^{-1}(s) f_x(s)\, \mathrm{d}s \right| \leq$$
$$\int_{t_0}^{t} |\psi^{-1}(t) Y(t) P_1 Y^{-1}(s)|\, \varphi(s) g(s, 2\varrho)\, \mathrm{d}s \leq$$
$$\leq |\psi^{-1}(t) Y(t) P_1| \cdot \int_{t_0}^{t} |P_1 Y^{-1}(s) \varphi(s) g(s, 2\varrho)|\, \mathrm{d}s .$$

Since (from Lemma 3)

$$|\psi^{-1}(t) Y(t) P_1| \in L_p(t_0, \infty)$$

and d) holds, it is evident that this first term belongs to $L_p(t_0, \infty)$. For the second term we have

$$\int_{t}^{\infty} |\psi^{-1}(t) Y(t) P_2 Y^{-1}(s)|\, |f_x(s)|\, \mathrm{d}s \leq$$
$$\leq \int_{t}^{\infty} |\psi^{-1}(t) Y(t) P_2 Y^{-1}(s)| \varphi(s) g(s, 2\varrho)\, \mathrm{d}s \leq$$
$$\leq \left(\int_{t}^{\infty} |\psi^{-1}(t) Y(t) P_2 Y^{-1}(s) \varphi(s)|^p\, \mathrm{d}s \right)^{1/p} \cdot \left(\int_{t}^{\infty} g^{p'}(s, 2\varrho)\, \mathrm{d}s \right)^{1/p'} \leq$$
$$\leq K \left(\int_{t}^{\infty} g^{p'}(s, 2\varrho)\, \mathrm{d}s \right)^{1/p'} .$$

Thus from b) (ii) and Lemma 2 we get that also this term belongs to $L_p(t_0, \infty)$. The proof of the theorem is complete.

Remark 1. If we substitute in Theorem 4 the condition b) (ii) by the condition

$$\left(\int_t^{\infty} g^{p'}(s, c)\, ds\right)^{1/p'} \in L_p(0, \infty)$$

and for p we assume that $1 < p < \infty$, then the conclusion of Theorem 4 holds.

Corollary 2. Let $p = 1$ (and $p' = \infty$). Assume that the assumptions of Theorem 4 are satisfied except b) (ii) which is substituted by conditions

$$\lim_{t \to \infty} \gamma_c(t) = 0 \quad \text{for each } c \geq 0$$

and

$$\gamma_c(t) \in L_1(0, \infty)\,,$$

where $\gamma_c(t) = \sup_{s \geq t} g(s, c)$.

Then the conclusion of Theorem 4 holds.

Corollary 3. Let $p = \infty$ (and $p' = 1$). Let condition a) of Theorem 4 be replaced by

$$\sup_{t_0 \leq s \leq t} |\psi^{-1}(t) Y(t) P_1 Y^{-1}(s) \varphi(s)| +$$
$$+ \sup_{t < s < \infty} |\psi^{-1}(t) Y(t) P_2 Y^{-1}(s) \varphi(s)| \leq K$$

and

$$|\psi^{-1}(t) Y(t) P_1| \in L_v(0, \infty),\ v > 1$$

and all the other assumptions of the Theorem 4 hold.

Then the sets of ψ-bounded solution of (1) and of (2) are (ψ, v)-integral equivalent.

In the same way as in Theorem 4 we can prove

Theorem 5. Assume that the following hypotheses from Theorem 4 are satisfied: a), b) (i), (iii). Instead of b) (ii) let

$$\int_0^{\infty} \varphi^p(t) \psi^{-p}(t)\, dt = \infty$$

be satisfied.

Finally, let the left side of the inequality a) belong to $L_1(0, \infty)$. Then the conclusions of Theorem 4 are still valid.

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SÚHRN

INTEGRÁLNA EKVIVALENCIA DIFERENCIÁLNYCH RELÁCIÍ

Alexander Haščák, Bratislava

V práci je zavedený pojem multifunkcie slabo polokompaktnej z hora. Využitie tohto pojmu umožňuje použiť Ky Fanovu vetu o pevnom bode. Jej využitím sú zovšeobecnené niektoré doterajšie výsledky o integrálnej ekvivalencii diferenciálnych systémov pre systémy tvaru: $x' \in F(t, x)$ a $y' = f(t, y)$.

РЕЗЮМЕ

ИНТЕГРАЛЬНАЯ ЭКВИВАЛЕНТНОСТЬ ДИФФЕРЕНЦИАЛЬНЫХ ВКЛЮЧЕНИЙ

Александер Хащак, Братислава

В работа вводится понятие мультифункции слабо полукомпактной сверху, использование которого дает нам возможность применить теорему Кы Фана о неподвижной точке. С ее помощью обобщены некоторые до этого времени известные результаты о интегральной эквивалентности дифференциальных систем для систем вида: $x' \in F(t, x)$ и $y' = f(t, y)$.