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ASYMPTOTIC BEHAVIOUR OF PERTURBED NONLINEAR SYSTEMS

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1. Introduction

We shall consider the nonlinear differential system

$$x' = f(t, x) \tag{1}$$

and the perturbed functional-differential system belonging to it

$$y' = f(t, y) + g(t, y, Ty_t) . \tag{2}$$

Here x, y are n-dimensional vectors. We make the following assumptions about f, g, T, x, y:

Assumption I. $f(t, y)$, $f_y(t, y) \in C[I \times D, R^n]$, where $I = [0, \infty)$, $D \subset R^n$ is a domain;

Assumption II. $g(t, y, z) \in C[I \times D \times E, R^n]$, where $E = C[[-h, 0], R^n]$, $h > 0$;

Assumption III. T is a continuous mapping E into itself such that for u_t, $v_t \in E$ the inequality $u_t \leqslant v_t$ (here and further we consider a natural ordering of E) implies $Tu_t \leqslant Tv_t$, where z_t denotes the element of E defined by $z_t(s) = z(t+s)$, $-h \leqslant s \leqslant 0$;

Assumption IV. $x(t, t_0, x_0)$ will denote the solution of (1) satisfying $x(t_0, t_0, x_0) = x_0$, $(t_0, x_0) \in I \times D$ such that for each $x_0 \in D$ it exists for any $t \geqslant t_0$.

The purpose of this paper is to study the asymptotic relationships between the solutions of (1) and (2) by means of an analogue of the classical variation of constants formula introduced by V. M. Alexeev [1] from which it follows that the equation (2) and the integro-differential equation

$$\begin{gathered} y_{t_0} = p \\ y(t) = x(t, t_0, p(0)) + \int_{t_0}^{t} F(t, s, y(s)) g(s, y(s), Ty_s)\, ds \end{gathered} \tag{3}$$

are mutually equivalent for $t \geq t_0$, where the derivative matrix $x_z(t, t_0, z) = F(t, t_0, z)$ satisfies the variational equation

$$w' = f_x(t, x(t, t_0, z))w ,$$

$F(t_0, t_0, z) =$ identity matrix and $x_{t_0}(t, t_0, z) = -F(t, t_0, z)f(t_0, z)$.

We begin by recalling some preliminary results which will be useful in the proofs of the main theorems. In the previous work [2] the following assertions have been proved:

Lemma 1. Let $p(t)$, $m(t) \in C[[t_0 - h, \infty), R_+]$, $G(t, u, z) \in C[I \times R_+ \times \times E_1, R_+]$, where $E_1 = C[[-h, 0], R_+]$ be nondecreasing in the last two variables for any $t \in I$. Let T_1 be a continuous mapping defined on E_1 into itself satisfying the Assumption III. Let $q \in E_1$ be an initial function and let $r(t) = r(t, t_0, q)$ be an arbitrary solution of

$$\begin{gathered} r_{t_0} = q \\ r(t) = p(t) + \int_{t_0}^{t} G(s, r(s), T_1 r_s)\, ds, \quad t \geq t_0 \end{gathered} \tag{4}$$

on the interval $[t_0 - h, c)$, $(c \leq +\infty)$. Finally, let

$$\begin{gathered} m_{t_0} < r_{t_0} \\ m(t) < p(t) + \int_{t_0}^{t} G(s, m(s), T_1 m_s)\, ds, \quad t \geq t_0 . \end{gathered} \tag{5}$$

Then the inequality

$$m(t) < r(t) \tag{6}$$

holds for every $t \in [t_0, c)$.

Lemma 2. Let the assumptions of Lemma 1 be true. Let $\bar{r}(t) = \bar{r}(t, t_0, q)$ be a maximal solution of (4) which exists in I. Let $k(t) \in C[[t_0 - h, \infty)R_+]$ be such that

$$\begin{gathered} k_{t_0} \leq q \\ k(t) \leq p(t) + \int_{t_0}^{t} G(s, k(s), T_1 k_s)\, ds, \quad t \geq t_0 . \end{gathered} \tag{7}$$

Then the inequality

$$k(t) \leq \bar{r}(t) \tag{8}$$

holds for every $t \geq t_0$.

Remark. The maximal solution of (4) is defined as follows: A solution $\bar{r}(t)$ is said to be the maximal solution of (4) if for any other solution $r(t)$ of (4) the inequality $r(t) \leq \bar{r}(t)$ holds on the common interval of existence.

2. **Main results.**

Theorem 1. Let the assumptions I.—IV. be true. Let $q \in E$ be such that $q(0) \in D$. Let the maximal solution $\bar{r}(t) = \bar{r}(t, t_0, |q(0)|)$ of the equation

$$r_t = |q(0)|$$
$$r(t) = |q(0)| + \int_{t_0}^{t} G(s, r(s), T_1 r_s)\, ds$$

exist for every $t \geq t_0$ and have finite limit $\lim_{t \to \infty} \bar{r}(t) = r_\infty$, where G and T_1 have the same properties as in Lemma 1 and $|.|$ denote a norm in R^n or the maximum norm in C, respectively. We denote by D^0 a compact subset of D and $Z = \{z \in C[I, D]\}$. Let $H(t)$ be a continuous nonsingular $n \times n$ matrix function for $t \geq t_0$ such that

(a) $t \geq t_0$, $z \in Z$, $|H(t)z(t)| \leq \bar{r}(t)$ implies $z(t) \in D^0$;
(b) $|H(t)F(t, s, z)g(s, z, Tz_s| \leq G(s, |H(s)z(s)|, T_1|Hz|_s)$ for $z \in Z$, $t, s \geq t_0$;
(c) $|H(t)x(t, t_0, z_0)| \leq |q(0)|$ for $z_0 \in D$, $t \geq t_0$, where $x(t, t_0, z_0)$ is a solution of (1);
(d) $t_0 \in I$, $t \geq t_0$, $z \in Z$, $z(t) \in D^0$ implies that $g(t, z, Tz_t)$ is a bounded function.
Then the solution $y(t, t_0, q)$ of (2) exists for every $t \geq t_0$ and for it there exists a corresponding solution $x^0(t)$, $t \geq t_0$ of (1) such that

$$\lim_{t \to \infty} H(t)[y(t) - x^0(t)] = 0\,. \tag{9}$$

Proof. The solution $y(t) = y(t, t_0, q)$ of (2) also fulfils (3) with $x(t_0) = q(0)$. Therefore we have

$$|H(t)y(t)| \leq |H(t)x(t)| + \int_{t_0}^{t} |H(t)F(t, s, y(s))g(s, y(s), Ty_s)|\, ds \leq$$
$$\leq |q(0)| + \int_{t_0}^{t} G(s, |H(s)y(s)|, T_1|Hy|_s)\, ds\,. \tag{10}$$

Applying Lemma 2 from the last inequality we obtain

$$|H(t)y(t)| \leq \bar{r}(t),\; t_0 \leq t < T \;\; (\leq +\infty)\,, \tag{11}$$

where $[t_0, T)$ is maximal interval of existence of the solution $y(t)$ of (2). Thus, by (a) in the assumptions we have $y(t) \in D^0$, therefore $g(t, y(t), Ty_t)$ is a bounded function. If $T < +\infty$, also the function $f(t, y(t))$ is bounded, so that by (2) $y'(t)$ is bounded for $t_0 \leq t < T$. This implies that there exists a finite limit $\lim_{t \to T} y(t)$, hence the solution $y(t)$ of (2) can be extended up to the point T and thus on the interval $[t_0, T + d)$, $d > 0$ which is a contradiction. Hence it follows that $T = +\infty$, so as $y(t)$ exists for every $t \geq t_0$ and it is a solution of (2); from the assumptions of Theorem it also follows that $\int_{t_0}^{\infty} G(s, \bar{r}(s), T_1\bar{r}_s)\, \mathrm{d}s < +\infty$. Thus, for an arbitrary $\varepsilon > 0$ there exists $u \geq t_0$ such that for each $t_1, t_2 \geq u$ $(t_1 < t_2)$ the inequality

$$\left| H(t) \int_{t_1}^{t_2} F(t, s, y(s)) g(s, y(s), Ty_s)\, \mathrm{d}s \right| \leq \int_{t_1}^{t_2} G(s, \bar{r}(s), T_1\bar{r}_s)\, \mathrm{d}s < \varepsilon$$

is satisfied. Therefore with respect to the regularity and continuity of $H(t)$ the improper integral

$$\int_{t_0}^{\infty} F(t, s, y(s)) g(s, y(s), Ty_s)\, \mathrm{d}s \tag{12}$$

converges uniformly for t on each compact subinterval from $[t_0, \infty)$.

From (3) we see that for $t \geq t_0$

$$H(t)y(t) = H(t)[x(t, t_0, q(0)) + \int_{t_0}^{\infty} F(t, s, y(s)) g(s, y(s), Ty_s)\, \mathrm{d}s] -$$
$$- H(t) \int_{t}^{\infty} F(t, s, y(s)) g(s, y(s), Ty_s)\, \mathrm{d}s$$

from where by (11) we obtain

$$|H(t)[y(t) - x(t, t_0, q(0)) - \int_{t_0}^{\infty} F(t, s, y(s)) g(s, y(s), Ty_s)\, \mathrm{d}s]| \leq$$
$$\leq \int_{t}^{\infty} G(s, \bar{r}(s), T_1\bar{r}_s)\, \mathrm{d}s = r_\infty - \bar{r}(t)\,.$$

Moreover, in accordance with the assumptions we have $\lim_{t\to\infty} [r_\infty - \bar{r}(t)] = 0$. Thus, the (9) is proved.

To finish the proof it remains to show that

$$x^0(t) = x(t, t_0, q(0)) + \int_{t_0}^{\infty} F(t, s, y(s)) g(s, y(s), Ty_s)\, \mathrm{d}s \tag{13}$$

is a solution of (1).

The relation $x(t, u, y(u)) + \int_{t_0}^{\infty} F(t, s, y(s)) g(s, y(s), Ty_s)\, \mathrm{d}s - x^0(t) = $
$= \int_{t_0}^{u} \frac{\mathrm{d}}{\mathrm{d}s} x(t, s, y(s))\, \mathrm{d}s$ is valid. Quite so $\frac{\mathrm{d}}{\mathrm{d}s} x(t, s, y(s)) =$
$= F(t, s, y(s))[y'(s) - f(s, y)] = F(t, s, y(s)) g(s, y(s), Ty_s)$

holds, so that we obtain

$$\int_{t_0}^{u} \frac{\mathrm{d}}{\mathrm{d}s} x(t, s, y(s))\, \mathrm{d}s = \int_{t_0}^{u} F(t, s, y(s)) g(s, y(s), Ty_s)\, \mathrm{d}s \tag{14}$$

from which with respect to (12) it follows

$$\lim_{u\to\infty} x(t, u, y(u)) = x^0(t)$$

uniformly for t on each compact subinterval from $[t_0, \infty)$. Because of

$$\frac{\mathrm{d}}{\mathrm{d}s} f[t, x(t, s, y(s))] = f_x[t, x(t, s, y(s))]F(t, s, y(s))[y'(s) - f(s, y(s))] = \\ = f_x[t, x(t, s, y(s))]F(t, s, y(s))g(s, y(s), Ty_s)$$

and

$$f(t, x^0(t)) - f[t, x(t, t_0, q(0))] = \lim_{u\to\infty} \int_{t_0}^{u} \frac{\mathrm{d}}{\mathrm{d}s} f[t, x(t, s, y(s))]\,\mathrm{d}s\,,$$

we have

$$f(t, x^0(t)) - f[t, x(t, x(t, t_0, q(0))] = \\ = \lim_{u\to\infty} \int_{t_0}^{u} f_x[t, x(t, s, y(s))]F(t, s, y(s))g(s, y(s) Ty_s)\,\mathrm{d}s$$

uniformly for t on each compact subinterval from $[t_0, \infty)$. Therefore, by differentiating (13) we obtain that

$$x^{0\prime}(t) = f[t, x(t, t_0, q(0))] + \\ + \int_{t_0}^{\infty} f_x[t, x(t, s, y(s))]F(t, s, y(s))g(s, y(s), Ty_s)\,\mathrm{d}s = \\ = f(t, x^0(t))$$

which completes the proof.

Theorem 2 below investigates the converse problem to that considered in Theorem 1.

Theorem 2. Let the assumptions of Theorem 1 hold. Then for any solution $x(t) = x(t, t_0, x_0)$ of (1) which exists for each $t \geq t_0$, where $x_0 \in D^0$, there is a $t_1 \geq t_0$ such that a solution $y(t)$ of (2) exists for every $t \geq t_1$ and the relation (9) remains valid.

Proof. We suppose that (2) does not reduce itself to (1). Then, let $0 < c < r_\infty - x_0$, choose $t_1 \geq t_0$ so large that $\int_{t_1}^{\infty} G(s, \bar{r}(s), T_1\bar{r}_s)\,\mathrm{d}s < c$. For any integer n such that $n \geqq t_1$ we define the set

$$Z_n = \{z \in C[[t_1, n], R^n] : |z(t) \leq |H(t)x(t)| + c\}\,.$$

First of all we show that under above conditions (2) has a solution $z_n(t)$ in Z_n.

In view of (3) and (9) it is suitable to define the operator S on Z_n as follows:

$$Sz(t) = H(t)x(t) + \int_{n}^{t} H(t)F(t, s, H^{-1}(s)z(s))g(s, H^{-1}(s)z(s), T_1(H^{-1}z)_s)\,\mathrm{d}s\,, \\ t_1 \leq t \leq n\,.$$

We notice that

$$|Sz(t)-H(t)x(t)|\leqslant\int_t^n G(s,|H^{-1}(s)z(s)|,T_1|H^{-1}z|_s)\,\mathrm{d}s\leqslant$$

$\leqslant\int_t^n G(s,\bar{r}(s),T_1\bar{r}_s)\,\mathrm{d}s<c$, thus $SZ_n\subset Z_n$. It is easy to see that all assumptions of the Schauder's fixed point theorem are fultilled. Thus, for any considered integer n there exists a fixed point $z_n\in Z_n$ such that $Sz_n=z_n$.

The function $y_n(t)=H^{-1}(t)z_n(t)$ satisfies

$$y_n'(t)=f(t,x(t))+g(t,y_n(t),Ty_{n_t})+$$
$$+\int_n^t f_x[t,x(t,s,y_n(s))]F(t,s,y_n(s))g(s,y_n(s),Ty_{n_s})\,\mathrm{d}s \tag{15}$$

and since

$$f[t,x(t,t,y_n(t))]-f(t,x(t))=\int_n^t \frac{\mathrm{d}}{\mathrm{d}s}f[t,x(t,s,y_n(s))]\,\mathrm{d}s=$$
$$=\int_n^t f_x[t,x(t,s,y_n(s))]F(t,s,y_n(s))[y_n'(s)-f(s,y_n(s))]\,\mathrm{d}s \tag{16}$$

we obtain

$$w(t)=-\int_n^t f_x[t,x(t,s,y_n(s))]F(t,O,y_n(s))w(s)\,\mathrm{d}s$$

where $w(t)=y_n'(t)-f(t,y_n(t))-g(t,y_n(t),Ty_{n_t})$. The last relation implies $w(t)=0$, thus $y_n(t)$ is the solution of (2) on $[t_1,n]$.

Let k be an integer larger than t_1. We consider the sequence $\{z_n\}$, $n=k$, $k+1,\ldots$ of fixed points obtained above. Clearly this sequence is uniformly bounded and equicontinuous on the interval $[t_1,k]$. By Ascoli—Arzela's theorem there is a subsequence $\{z_{n_1}\}$ which is for $n_1=k+1$ defined on $[t_1,k+1]$ and is uniformly bounded and equicontinuous.

Proceeding inductively we obtain a function $z(t)$ on $[t_1,\infty)$ and a chain of subsequences $\{z_{n_k}\}$ that $\{z_{n_k}\}$ converges uniformly to z on every $[t_1,n+k]$. Subsequence $\{z_{n_n}\}$ obtained by the diagonal choice converges uniformly to $z(t)$ on any compact subinterval of $[t_1,\infty)$.

Because $y_{n_n}(t)=H^{-1}(t)z_{n_n}(t)$ are solutions of (2), the limit function $y(t)=H^{-1}(t)z(t)$ is also a solution of (2) on $[t_1,\infty)$. With respect to the assumptions it is easy to see that the improper integral

$$\int_t^\infty H(t)F(t,s,H^{-1}(s)z(s))g(s,H^{-1}(s)z(s),T(H^{-1}z)_s)\,\mathrm{d}s$$

converges and since it is equal to $H(t)[y(t)-x(t)]$ we conclude that (9) is fulfilled, which completes the proof.

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SÚHRN

ASYMPTOTICKÉ CHOVANIE PERTURBOVANÝCH NELINEÁRNYCH SYSTÉMOV

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V práci sa vyšetruje nelineárny diferenciálny systém (1) $x' = f(t, x)$ a perturbovaný funkcionálno-diferenciálny systém (2) $y' = f(t, y) + g(t, y, Ty_t)$.

Skúmajú sa asymptotické vzťahy medzi riešeniami (1) a (2). Sú stanované podmienky existencie takého riešenia $y(t)$ (2) a k nemu prislúchajúceho riešenia $x(t)$ (1), že platí (3) $\lim_{t\to\infty} H(t)[y(t) - x(t)] = 0$, kde $H(t)$ je regulárna matica daných vlastností a tiež podmienky pre existenciu riešení obráteného problému pri zachovaní (3).

РЕЗЮМЕ

АСИМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ ВОЗМЫЩЕННЫХ НЕЛИНЕЙНЫХ СИСТЕМ

Владислав Роса, Братислава

В статье изучается нелинейная дифференциальная система (1) $x' = f(t, x)$ и возмущенная функциональнодифф. система (2) $y' = f(t, y) + g(t, y; Ty_t)$. Исследовано асимптотическое

поведение решений 1) и (2). Определены условия для существования такого решения $y(t)$ (2) и соответствующего решения $x(t)$ (1), что (3)

$$\lim_{t\to\infty} H(t)[y(t)-x(t)]=0 ,$$

где $H(t)$ — данная невырожденная матрица. Также установлены условия для существования решений задачи, получающейся перестановкой $x(t)$, $y(t)$, при сохранении (3).