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THE EXISTENCE AND STABILITY OF CERTAIN FUNCTIONAL-DIFFERENTIAL SYSTEM

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In this paper a functional-differential equation will be investigated. It can be considered to be a perturbation of some ordinary nonlinear differential equation. The existence theorem will be proved and some sufficient conditions for the stability of solutions will be formed.

Let D be a region in R^n, let f be a continuous function from $I=[0, \infty)\times D$ into R^n such that $f_y(t, y)$ exists and is continuous on $I\times D$, let g be a continuous function from $I\times D\times E$ into R^n, where E is a family of continuous functions on $[-h, 0]$ for some $h>0$ and let T be a continuous mapping defined on E into itself such that for $u_t, v_t\in E$ the inequality $u_t\leqslant v_t$ implies $Tu_t\leqslant Tv_t$. (Here and further we consider the natural ordering of E). For a continuous function $y(t)$ defined for $t\geqslant -h$ and each $h>0$, y_t denotes the element of E defined by $y_t(s)=y(t+s)$, $-h\leqslant s\leqslant 0$.

We consider a nonlinear differential system

$$\frac{dx}{dt}=f(t, x) \tag{1}$$

and a perturbed functional-differential system

$$\frac{dy}{dt}=f(t, y)+g(t, y, Ty_t)\,. \tag{2}$$

Let $x(t, t_0, x_0)$ denote the solution of (1) passing through the point $(t_0, x_0)\in[0, \infty)\times D$. It will be always assumed that for arbitrary $t_0\geqslant 0$ and $x_0\in D$ the solution $x(t, t_0, x_0)$ of (1) exists (i.e. is defined) for every $t\geqslant t_0$.

We know [1] that under above conditions the matrix $F(t, t_0, x_0)=\dfrac{\partial x(t, t_0, x_0)}{\partial x_0}$ is the fundamental matrix solution of the variational system

$$u'=f_x[t, x(t, t_0, x_0)]u \tag{3}$$

such that $F(t_0, t_0, x_0)$ is the identity matrix.

A solution $y(t)$ of (2) given by an initial function $p \in E$ for $t=t_0$ we shall denote $y(t, t_0, p)$. Vertical bars will denote any appropriate vector and compatibile matrix norms, respectively. For every $z \in E$ we define norm by $\|z\| = \sup_{-h \leq s \leq 0} |z(s)|$.

Let D_1 be a convex subset of D. For any t for which there exist $x(t, t_0, x_0)$, $x(t, t_0, y_0)$ respectively, where $x_0, y_o \in D_1$ by [2] the inequality

$$|x(t, t_0, x_0) - x(t, t_0, y_0)| \leq |y_0 - x_0| \sup_{v \in D} |F(t, t_0, v)| \tag{4}$$

holds.

Let $\bar{E}$ be a family of continuous functions on $[-h, 0]$ into D. If $p \in \bar{E}$ then the solution $y(t, t_0, p)$ of (2) satisfies by [4] the functional-integral equation

$$\begin{gathered} y_{t_0} = p \\ y(t) = x(t, t_0, p(0)) + \int_{t_0}^{t} F(t, s, y(s)) g(s, y(s), Ty_s)\, ds \end{gathered} \tag{5}$$

for any $t \geq t_0$ such that $x(t, t_0, p(0))$, $y(t, t_0, p)$ exist.

Combination of (4) and (5) yields our first preliminary result.

Lemma 1. Let $x_0 \in D_1$, $p \in \bar{E}$, $p(0) \in D_1$. Then for every $t \geq t_0$ for which there exist $x(t, t_0, x_0)$ and $y(t, t_0, p)$ respectively, we have

$$\begin{aligned} |y(t, t_0, p) - x(t, t_0, x_0)| \leq |p(0) - x_0| \sup_{v \in D} |F(t, t_0, v)| + \\ + \int_{t_0}^{t} |F(t, s, y(s, t_0, p))|\, |g(s, y(s, t_0, p), Ty_s)|\, ds\,. \end{aligned} \tag{6}$$

Now we give some lemmas which embody suitable conclusions for the investigation of the relations between the solutions of (1) and (2) and which will be useful in the proof of main theorems below.

Lemma 2. Let $p(t)$ be a continuous function from $[t_0 - h, \infty)$ into R_+ and $G(t, u, z)$ be a continuous function from $I \times R_+ \times E_1$ into R_+, where $E_1 \subset E$ is a space of continuous functions from $[-h, 0]$ into R_+. Let T_1 be a continuous mapping defined from E_1 into itself such that for $u_t, v_t \in E_1$ the inequality $u_t \leq v_t$ implies $T_1 u_t \leq T_1 v_t$. Let $q \in E_1$ be an initial function.

Then there exists a positive number d such that the initial value problem

$$\begin{gathered} r_{t_0} = q \\ r(t) = p(t) + \int_{t_0}^{t} G(s, r(s), T_1 r_s)\, ds, \quad t \geq t_0 \end{gathered} \tag{7}$$

where $q(0) = p(t_0)$, has a solution $r(t, t_0, q)$ on the interval $[t_0 - h, t_0 + d]$.

Proof. Let a, b, M be such positive constants that for

$$t_0 \leqslant t \leqslant t_0 + a,\ p(t) \leqslant r(t) \leqslant p(t) + b,\ \bar{p}_t \leqq r_t \leqslant \bar{p}_t + b\ ,$$

where

$$\bar{p}(t) = \begin{cases} q(t - t_0), & t_0 - h \leqslant t \leqslant t_0 \\ p(t), & t_0 \leqslant t \leqslant t_0 + a \end{cases}$$

the inequality

$$G(t, r, T_1 r_t) \leqslant M$$

holds. We denote by B the space of continuous functions from $[t_0 - h, t_0 + d]$ into R_+, where $d = \min(a, b/M)$. If for any $r \in B$ we define norm $|.|_B$ by relation $|r|_B = \sup\limits_{t_0 - h \leqq t \leqq t_0 + d} r(t)$, B is a Banach space.

We define a set $S \subset B$ as follows:

$$S = \left\{ r \in B\colon \begin{array}{l} r(t) = q(t - t_0), \\ \left. \begin{array}{l} p(t) \leqslant r(t) \leqslant p(t) + b, \\ p_t \leqslant r_t \leqslant p_t + b, \end{array} \right\} \end{array} \begin{array}{l} t_0 - h \leqslant t \leqslant t_0 \\ t_0 \leqslant t \leqslant t_0 + d \end{array} \right\}$$

It is easy to see that S is a convex and closed set. Now we define an operator Z on S by $Zr = q$, where

$$\begin{aligned} q(t) &= q(t - t_0),\ t_0 - h \leqslant t \leqslant t_0 \\ q(t) &= p(t) + \int_{t_0}^{t} G(s, r(s), T_1 r_s)\, \mathrm{d}s,\ t_0 \leqslant t \leqslant t_0 + d\ . \end{aligned} \tag{8}$$

Obviously $p(t) \leqslant Zr(t) \leqslant p(t) + Md \leqslant p(t) + b$, so that $ZS \subset S$. We have

$$Zr_1(t) - Zr_2(t) = 0,\ t_0 - h \leqslant t \leqslant t_0$$

$$\begin{aligned} |Zr_1(t) - Zr_2(t)| &= \left| \int_{t_0}^{t} [G(s, r_1(s), T_1 r_{1s}) - G(s, r_2(s), T_1 r_{2s})]\, \mathrm{d}s \right| \leqslant \\ &\leqslant \int_{t_0}^{t} |G(s, r_1(s), T_1 r_{1s}) - G(s, r_2(s), T_1 r_{2s})|\, \mathrm{d}s \leqslant \\ &\leqslant \int_{t_0}^{t_0 + d} |G(s, r_1(s), T_1 r_{1s}) - G(s, r_2(s), T_1 r_{2s})|\, \mathrm{d}s,\ t \geqslant t_0\ . \end{aligned} \tag{9}$$

Since the function G and operator T_1 are continuous, it follows that for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for $t_0 \leqslant t \leqslant t_0 + d$, $r_1, r_2 \in S$ the inequalities $|r_1(t) - r_2(t)|_B < \delta$, $|r_{1t} - r_{2t}|_1 < \delta$ imply

$$|G(t, r_1(t), T_1 r_{1t}) - G(t, r_2(t), T_1 r_{2t})| < \frac{\varepsilon}{d}\ .$$

Hence by (9) $|Zr_1(t) - Zr_2(t)|_B < \varepsilon$, thus the operator Z is continuous on S. Here $|.|_1$ denotes a norm for the space E_1. Further for $t_1, t_2 \in [t_0, t_0 + d]$

$$|Zr(t_1) - Zr(t_2)| \leq |p(t_1) - p(t_2)| + \left| \int_{t_1}^{t_2} G(s, r(s), T_1 r_s)\, ds \right| \leq$$
$$\leq |p(t_1) - p(t_2)| + M|t_2 - t_1| .$$

Since p is the continuous function, for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for $|t_1 - t_2| < \delta$ the inequality $|p(t_1) - p(t_2)| < \frac{\varepsilon}{2}$ holds. If we take $\delta = \frac{\varepsilon}{2M}$ we obtain $|Zr(t_1) - Zr(t_2)| < \frac{\varepsilon}{2} + M \frac{\varepsilon}{2M} = \varepsilon$, thus the set ZS is equicontinuous. Since ZS is also uniformly bounded, applying the Schauder fixed point theorem it follows that there exists at least one function $\bar{r} \in S$ such that $Z\bar{r} = \bar{r}$, $t_0 - h \leq t \leq t_0 + d$ from which we have

$$\bar{r}(t) = q(t - t_0), \quad t_0 - h \leq t \leq t_0$$
$$\bar{r}(t) = p(t) + \int_{t_0}^{t} G(s, \bar{r}(s), T_1 \bar{r}_s)\, ds, \quad t_0 \leq t \leq t_0 + d$$

and therefore $\bar{r}(t, t_0, q)$ is the solution of (7) on $[t_0 - h, t_0 + d]$, which completes the proof.

Remark 1. If there exists a solution of the integral equation (7) on $[t_0 - h, t_0 + d]$, by a well-known property of the Volterra integral equations this solution can be continued. Indeed, for $t \geq t_0 + d$ we can write the solution in the form

$$r(t) = \left[p(t) + \int_{t_0}^{t_0 + d} G(s, r(s), T_1 r_s)\, ds \right] + \int_{t_0 + d}^{t} G(s, r(s), T_1 r_s)\, ds,$$

where in the square brackets the known function is situated and then continue solving the equation. From this some important properties of the solution $r(t)$ of (7) follow, first of all:

i) the solution can be continued on some maximal interval of its existence $[t_0 - h, c)$;

ii) if $[t_0 - h, c)$ is the maximal interval of existence of the solution, then there cannot exist a proper (finite) limit $\lim_{t \to c^-} r(t)$.

Lemma 3. Let the assumptions of Lemma 2 be satisfied. Let G be a nondecreasing function in last two variables for each $t \in I$. Let m be a continuous function from $[t_0 - h, \infty)$ into R_+ and let $r(t) = r(t, t_0, q)$ be an arbitrary solution of (7) existing on $[t_0 - h, c)$, $(c \leq +\infty)$. Then, providing

$$\begin{gathered} m_{t_0} < r_{t_0} \\ m(t) < p(t) + \int_{t_0}^{t} G(s, m(s), T_1 m_s)\, ds, \quad t \geq t_0 \end{gathered} \tag{10}$$

the inequality

$$m(t)<r(t) \tag{11}$$

holds for each $t\in[t_0, c)$.

Proof. Suppose the opposite. Then the set $W=\{t\in[t_0, c): m(t)\geqslant r(t)\}$ is nonempty. We denote $t_1=\inf W$. Obviously $t_1>t_0$ and $m(t)<r(t)$, $t_0\leqslant t<t_1$, $m(t_1)=r(t_1)$. Because of $m_{t_0}<r_{t_0}$ it follows $m_t<r_t$ for $t_0\leqslant t\leqslant t_1$ and since T_1 is a nondecreasing operator, for $t_0\leqslant t\leqslant t_1$ we have $T_1m_t\leqslant T_1r_t$. Also, since G is monotonous, for $t_0\leqslant t\leqslant t_1$ we obtain $G(t, m(t), T_1m_t)\leqslant G(t, r(t), T_1r_t)$. However by (7) and (8) we have $m(t_1)<p(t_1)+\int_{t_0}^{t_1} G(s, m(s), T_1m_s)\,\mathrm{d}s\leqslant p(t_1)+\int_{t_0}^{t_1} G(s, r(s), T_1r_s)\,\mathrm{d}s = r(t_1)$, which is contradictory to the assumption $m(t_1)=r(t_1)$. Thus, the set W is empty and the assertion is proved.

Definition 1. A solution $\bar{r}(t)$ $[\underline{r}(t)]$ is said to be a maximal [minimal] solution of the (7) if for any other solution $r(t)$ of (7) the inequality $r(t)\leqslant\bar{r}(t)$ $[r(t)\geqslant\underline{r}(t)]$ holds on the common interval of existence.

Lemma 4. Let the assumptions of Lemma 3 be satisfied. Let $q\in E_1$ be a given initial function. Then there exists a positive number d such that the maximal solution of (7) exists on $[t_0-h, t_0+d]$.

Proof. Procceding like in the previous lemma we obtain $a>0$, $b>0$, $M>0$. Since G is continuous on $[t_0, t_0+a]\times[p(t), p(t)+b]\times[\bar{r}(t), \bar{r}(t)+b]$ and T_1 is continuous on E_1, we obtain $G(t, r, T_1r_t)<M+\frac{b}{2}$ for $t_0\leqslant t\leqslant t+a$, $p(t)\leqslant r\leqslant p(t)+b+\varepsilon$, $\bar{p}_t\leqslant r_t\leqslant\bar{p}_t+b+\varepsilon$, and for each $\varepsilon\in(0, \varepsilon_0)$.

We consider the initial value problem

$$r_{t_0}=q+\varepsilon$$

$$r(t)=p(t)+\varepsilon+\int_{t_0}^{t} G(s, r(s), T_1r_s)\,\mathrm{d}s, \quad t\geqslant t_0 \tag{12}$$

$$p(t_0)=q(0),$$

where $\varepsilon\in(0, \varepsilon_0)$.

Using Lemma 2 we obtain that this problem has a solution $r(t, t_0, q, \varepsilon)$ which exists on $[t_0-h, t_0+d]$, where

$$d=\min\left(a, \frac{2b}{2M+b}\right).$$

For $0<\varepsilon_2<\varepsilon_1\leqslant\varepsilon$ the inequalities

$$r_{t_0}(t_0, q, \varepsilon_2)<r_{t_0}(t_0, q, \varepsilon_1)$$

$$r(t, t_0, q, \varepsilon_2)<p(t)+\varepsilon_1+\int_{t_0}^{t} G(s, r(s, t_0, q, \varepsilon_2), T_1r(t_0, q, \varepsilon_2)_s)\,\mathrm{d}s$$

hold for $t_0 \leq t \leq t_0 + d$ from which using Lemma 3 we have

$$r(t, t_0, q, \varepsilon_2) < r(t, t_0, q, \varepsilon_1), \quad t_0 \leq t \leq t_0 + d\,.$$

Since the system $r(t, t_0, q, \varepsilon)$ is equicontinuous and uniformly bounded on $[t_0, t_0 + d]$ and the system $T_1 r(t_0, q)_t$ is compact on E_1, applying the Ascoli—Arzelà theorem there exists a monotonous, decreasing sequence $\{\varepsilon_n\}$ which converges to zero as n tends to $+\infty$ such that $\lim_{n \to \infty} r(t, t_0, q, \varepsilon_n) = r(t, t_0, q, 0) = r^0(t, t_0, q)$ and $\lim_{n \to \infty} T_1 r(t_0, q, \varepsilon_n)_t = T_1 r(t_0, q, 0)_t = T_1 r^0(t_0, q)_t$ uniformly for $t \in [t_0, t_0 + d]$. Also, since G is uniformly continuous, we obtain that $G[t, r(t, t_0, q, \varepsilon_n), T_1 r(t_0, q, \varepsilon_n)_t]$ converges uniformly to $G[t, r^0(t, t_0, q), T_1 r^0(t_0, q)_t]$ as n tends to $+\infty$ so that the convergence principle can be used in the equation $r(t, t_0, q, \varepsilon_n) = p(t) + \varepsilon_n + \int_{t_0}^{t} G(s, r(s, t_0, \varepsilon_n), T_1 r(t_0, q, \varepsilon_n)_s)\, ds$ $t_0 \leq t \leq t_0 + d$. Evidently $r^0(t_0, q)_{t_0} = q$, so the limit function $r^0(t, t_0, q)$ is the solution of (7) on $[t_0 - h, t_0 + d]$.

To finish the proof it remains to prove that $r^0(t, t_0, q)$ is the maximal solution of (7). Indeed, let $r(t, t_0, q)$ be a solution of (7) existing on $[t_0 - h, t_0 + d]$. For any $\varepsilon \in (0, \varepsilon_0)$ the inequalities

$$r_{t_0}(t_0, q) < r_{t_0}(t_0, q, \varepsilon)$$

$$r(t, t_0, q) < p(t) + \varepsilon + \int_{t_0}^{t} G(s, t_0, q), T_1 r(t_0, q)_s)\, ds$$

and the equality

$$r(t, t_0, q, \varepsilon) = p(t) + \varepsilon + \int_{t_0}^{t} G(s, r(s, t_0, q, \varepsilon), T_1 r_s(t_0, q, \varepsilon))\, ds$$

for $t_0 \leq t \leq t_0 + d$ hold. Hence by Lemma 3 for the same t the estimation $r(t, t_0, q) < r(t, t_0, q, \varepsilon)$ holds. Also, because $\lim_{\varepsilon \to 0^+} r(t, t_0, q, \varepsilon) = r^0(t, t_0, q)$ uniformly on $[t_0, t_0 + d]$ we obtain that on the whole interval of existence the inequality $r(t, t_0, q) \leq r^0(t, t_0, q)$ is fulfilled, which completes the proof.

Remark 2. Analogically the existence of a minimal solution of (7) can be proved.

Remark 3. Both the maximal or the minimal solution of (7) respectively can be continued (as a maximal or minimal solution respectively) on the interval $[t_0 - h, c)$.

Lemma 5. Let the assumptions of Lemma 4 be satisfied. Let $\bar{r}(t) = \bar{r}(t, t_0, q)$

be a maximal solution of (7) which exists on I. Let $k(t)$ be a continuous function from $[t_0-h, \infty)$ into R_+ such that

$$\begin{gathered} k_{t_0} \leqslant q \\ k(t) \leqslant p(t) + \int_{t_0}^{t} G(s, k(s), T_1 k_s)\, \mathrm{d}s, \quad t \geqslant t_0 \,. \end{gathered} \tag{13}$$

Then the estimate

$$k(t) \leqslant \bar{r}(t) \tag{14}$$

is true for each $t \geqslant t_0$.

Proof. To employ Lemma 3 and Lemma 4 we need only to show that for any compact subinterval there is an $\varepsilon_0 > 0$ such that the solution $r(t, \varepsilon)$ exists on it for every $\varepsilon \in [0, \varepsilon_0)$ and make the considerations on such a one compact subinterval only. The proof of this property can be realized analogically to the proof of Theorem 3.2 in [3]. After having found a solution $r(t, \varepsilon)$ for every $t \geqslant t_0$ and any $\varepsilon \in [O, \varepsilon_0)$ we have to ascertain that for all considered t and $\varepsilon > 0$ the estimate

$$k(t) < r(t, \varepsilon) \tag{15}$$

is fulfilled. This relation can be proved similarly as in Lemma 3. In accordance with Lemma 4 we have $\lim\limits_{\varepsilon \to 0^+} r(t, \varepsilon) = \bar{r}(t)$ uniformly in t, which completes the proof.

Definition 2. The system (2) is said to be stable if for an arbitrary $\varepsilon > 0$ and $t_0 \geqslant 0$ there exists $\delta > 0$ such that for every two solutions $y(t, t_0, p_1)$, $\bar{y}(t, t_0, p_2)$ the $|y(t, t_0, p_1) - \bar{y}(t, t_0, p_2)| < \varepsilon$ holds for every $t \geqslant t_0$ and every $p_1, p_2 \in E$ such that $\|p_1 - p_2\| < \delta$.

In addition, if $\lim\limits_{t \to \infty} |y(t, t_0, p_1) - \bar{y}(t, t_0, p_2)| = 0$ the system (2) is said to be asymptotic stable.

Remark 4. The above definition can be applied to the system (1), too.

Definition 3. The equation (7) is said to be stable if for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for the maximal solution $\bar{r}(t) = \bar{r}(t, t_0, q)$, $\bar{r}(t) < \varepsilon$ holds for every $t \geqslant t_0$, every function $p(t) < \delta$ and every initial function q such that $\|q\| < \delta$.

In addition, if $\lim\limits_{t \to \infty} \bar{r}(t) = 0$, the equation (7) is said to be asymptotic stable.

Theorem 1. Let for the function $g(t, y, Ty_t)$ the estimate

$$\begin{gathered} |F(t, s, y(s))g(s, y(s), Ty_s) - F(t, s, \bar{y}(s))g(s, \bar{y}(s), T\bar{y}_s| \leqslant \\ \leqslant G(s, |y(s) - \bar{y}(s)|, T_1|y - \bar{y}|_s), \quad t, s \in I \end{gathered} \tag{16}$$

be fulfilled, where the function G and the operator T_1 are the same as in Lemma 2. Let $x(t)$, $\bar{x}(t)$ be some solutions of (1). Then the stability [asymptotic stability] of

(1) and (7), where $p(t)=|x(t)-\bar{x}(t)|$ implies the stability [asymptotic stability] of the perturbed system (2), respectively.

Proof. Because of the stability of (1), for $|x_0-\bar{x}_0|<\gamma$ we have $|x(t)-\bar{x}(t)|<$ $<\delta$. Similarly, from the stability of (7) for $\|q\|<\delta$, $p(t)<\delta$ we have $\bar{r}(t, t_0, \|q\|)<\varepsilon$, all for $t\geqslant t_0$.

We shall show that for $|r_1-r_2|_B<\delta$ the inequality $|y(t, t_0, r_1)-\bar{y}(t, t_0, r_2)|<\varepsilon$ holds for every $t\geqslant t_0$. Suppose a contrary. Then there exist solutions $y(t, t_0, r_1)$, $\bar{y}(t, t_0, r_2)$ and $t_1>t_0$ such that $|y(t_1)-\bar{y}(t_1)|=\varepsilon$, $|y(t)-\bar{y}(t)|<\varepsilon$, $t_0\leqslant t<t_1$.

From (5) we obtain

$$\begin{gathered} y_{t_0}=r_1 \\ y(t)=x(t, t_0, r_1(0))+\int_{t_0}^{t} F(t, s, y(s))g(s, y(s), Ty_s)\,\mathrm{d}s, \quad t\geqslant t_0\,. \end{gathered} \tag{17}$$

Similarly for the solution $\bar{y}(t, t_0, t_2)$ of the problem

$$\begin{gathered} \bar{y}_{t_0}=r_2 \\ \frac{d\bar{y}}{dt}=f(t, \bar{y})+g(t, \bar{y}, T\bar{y}_t) \end{gathered} \tag{2'}$$

we obtain

$$\begin{gathered} \bar{y}_{t_0}=r_2 \\ \bar{y}(t)=\bar{x}(t, t_0, r_2(0))+\int_{t_0}^{t} F(t, s, \bar{y}(s))g(s, \bar{y}(s), T\bar{y}_s)\,\mathrm{d}s, \quad t\geqslant t_0\,. \end{gathered} \tag{18}$$

We have

$$\begin{gathered} y_{t_0}-\bar{y}_{t_0}=r_1-r_2 \\ y(t)-\bar{y}(t)=x(t)-\bar{x}(t)+\int_{t_0}^{t} [F(t, s, y(s))g(s, y(s), Ty_s)- \\ -F(t, s, \bar{y}(s))g(s, \bar{y}(s), T\bar{y}_s)]\,\mathrm{d}s\,. \end{gathered}$$

Consequently, if we denote $|r_1-r_2|_B=r$, $|y(t)-\bar{y}(t)|=m(t)$, $|x(t)-\bar{x}(t)|=p(t)$, using the (16) we obtain

$$\begin{gathered} m_t\leqslant|r|_B \\ m(t)\leqslant p(t)+\int_{t_0}^{t} G(s, m(s), T_1m_s)\,\mathrm{d}s \\ p(t_0)=|r_1(0)-r_2(0)|=|r(0)|\,. \end{gathered}$$

Thus, applying Lemma 5 we obtain

$$m(t)=|y(t)-\bar{y}(t)|\leqslant\bar{r}(t, t_0, |r|_B), \quad t\geqslant t_0\,. \tag{19}$$

For $t = t_1$ we have

$$\varepsilon = m(t_1) = |y(t_1) - \bar{y}(t_1)| \leqslant \bar{r}(t_1, t_0, |r|_B) < \varepsilon$$

which is a contradiction. Hence the inequality $|y(t) - \bar{y}(t)| < \varepsilon$ is valid for every $t \geqslant t_0$, and thus the stability of (2) is proved.

If we assume the asymptotic stability of (1) and (7) we obtain as a consequence of the achieved considerations that the system (2) is stable so that the (19) is true for every $t \geqslant t_0$. From this it follows immediately that if $\lim_{t\to\infty} \bar{r}(t) = 0$ then also $\lim_{t\to\infty} |y(t) - \bar{y}(t)| = 0$, which completes the proof of the theorem.

In the paper [5] the following assertion has been formulated:

Lemma 6. Let $u(t)$, $f(t)$, $g(t)$ be continuous functions from I into R_+ such that

$$u(t) \leqslant u_0 + \int_{t_0}^{t} f(s)[u(s) + \int_{t_0}^{s} g(z)u(z)\,\mathrm{d}z]\,\mathrm{d}s, \quad t \geqq t_0, \tag{20}$$

where u_0 is a positive constant.

Then the estimate

$$u(t) \leqslant u_0[1 + \int_{t_0}^{t} f(s) \exp\left\{\int_{t_0}^{s} [\mathrm{f(z)} + \mathrm{g(z)}]\,\mathrm{dz}\right\} \mathrm{d}s] \tag{21}$$

holds for every $t \geqslant t_0$.

Definition 4. The trivial solution $x = 0$ of (1) is said to be exponential asymptotic stable if for every $t_0 \geqq 0$ there exist constants $K > 0$, $c > 0$ such that for any $t \geqslant t_0$ and $|x_0|$ sufficiently small the inequality

$$|x(t, t_0, x_0)| \leqslant K|x_0| \exp[-c(t - t_0)] \tag{22}$$

is fulfilled.

Theorem 2. Let the solution $x = 0$ of (1) be exponential asymptotic stable. Let $k(t)$, $m(t)$, $n(t)$ be continuous functions from I into R_+ which fulfil

$$\begin{gathered}\int_{t_0}^{\infty} m(t)\,\mathrm{d}t < \infty, \quad \int_{t_0}^{\infty} n(t)\,\mathrm{d}t < \infty, \\ k(t) \leqslant \exp[-c_1 t] \int_{t_0}^{t_0} n(s)|y(s)|\,\mathrm{d}s,\end{gathered} \tag{23}$$

where c_1 is a positive constant. Let for $(t, y, q) \in I \times D \times E$ the inequality

$$|g(t, y, q)| \leqslant m(t)[|y| + k(t)] \tag{24}$$

be true.

Then all solutions of (2) converge to zero as t tends to $+\infty$.

Proof. First of all we shall show that the solution $u_0=0$ of (3), where $f(t, x(t, t_0, x_0))=f(t, 0)$ is exponential asymptotic stable. Indeed, let $u(t)$ be a solution of (3) with the same initial condition as the solution $u_0(t)$ the equation

$$u_0'=f_x(t, 0)u_0 .$$

Fundamental matrix of this solution is $F(t, t_0, 0)=\dfrac{\partial x(t, t_0, 0)}{\partial x_0}$. For the components of the vector x_0 we have

$$\left|\frac{\partial x(t, t_0, 0)}{\partial x_0}\right| = \left|\lim_{h\to 0} \frac{x(t, t_0, x_{i0})-x(t, t_0, 0)}{h}\right| \leqq \lim_{h\to 0} K \exp[-c(t-t_0)] =$$

$$=K \exp[-c(t-t_0)],\ i=1, \dots, n ,$$

where x_{i0} is a vector which contains exactly unique non-zero i-th component whose value is equal to h, since $x(t, t_0, 0)=0$ and $|x(t, t_0, x_0)| \leq K|x_0| \exp[-c(t-t_0)]$. Hence

$$|F(t, t_0, 0)|\leq K \exp[-c(t-t_0)]$$

thus the solution $u_0=0$ is exponential asymptotic stable. Now, proceeding analogically as in the proof of Theorem 1 of [6] it can be shown that in a suitable domain D_1 the estimate

$$|F(t, t_0, x_0)|\leq K_1 \exp[-c_1(t-t_0)],\ t\geq t_0,\ K_1>0,\ c_1>0 \tag{25}$$

holds. If we assume that $D^0\subset D_1$ is a convex subset, $x_0\in D^0$, $q(0)\in D^0$, then from (23), (24) and (25) using Lemma 1, (where $x(t, t_0, 0)=0$) we obtain the estimate

$$|y(t)|\leq|q(0)|\, K_1 \exp[-c_1(t-t_0)]+$$
$$+\int_{t_0}^{t} K_1 \exp[-c(t-s)]m(s)\times[|y(s)|+\exp[-c_1 s] \int_{t_0}^{s} n(z)|y(z)|\, dz]\, ds . \tag{26}$$

Upon multiplying by $\exp[c_1 t]$ we obtain

$$|y(t)| \exp[c_1 t]\leq|q(0)|\, K_1 \exp[c_1 t_0]+\int_{t_0}^{t} K_1 \exp[c_1 s]m(s)[|y(s)|]+$$
$$+\exp[-c_1 s] \int_{t_0}^{s} n(z)|y(z)|\, dz]\, ds\leqq|q(0)|K_1 \exp[c_1 t_0]+\int_{t_0}^{t} K_1 m(s)[|y(s)\times$$
$$\times| \exp[c_1 s]+$$
$$+\int_{t_0}^{s} n(z)|y(z)|\, dz]\, ds\leq|q(0)|\, K_1 \exp[c_1 t_0]+\int_{t_0}^{t} K_1 m(s)[|y(s)| \exp[c_1 s]+$$
$$+\int_{t_0}^{s} n(z)|y(z)| \exp[c_1 z]\, dz]\, ds .$$

If we denote $|y(t)| \exp[c_1 t] = u(t)$, using Lemma 6 the last inequality leads to

$$|y(t) \exp[c_1 t] \leqq$$

$$\leqq |q(0)| K_1 \exp[c_1 t_0][1 + \int_{t_0}^{t} K_1 m(s) \exp\left[\int_{t_0}^{s} [K_1 m(z) + n(z)] \, dz\right] ds\right],$$

so that multiplying by $\exp[-c_1 t]$ finally we obtain

$$|y(t)| \leqq |q(0)| K_1 \exp[-c_1(t - t_0)][1 + \int_{t_0}^{t} K_1 m(s) \exp\left[\int_{t_0}^{s} [K_1 m(z) + n(z)] \, dz\right] ds], \quad t \geqq t_0 \tag{27}$$

for every solution $y(t)$ of (2).

Choose $q(0)$ and K_1 sufficiently small; from (27) it follows the assertion of the theorem. The proof is complete.

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SÚHRN

EXISTENCIA A STABILITA RIEŠENÍ ISTÉHO FUNKCIONÁLNO-DIFERENCIÁLNEHO SYSTÉMU

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V práci sa skúma funkcionálno-diferenciálny systém, ktorý vznikne perturbáciou istej nelineárnej diferenciálnej rovnice. Sú dokázané vety o existencii riešenia systému (aj maximálneho a minimálneho riešenia) a sformulované postačujúce podmienky stability, asymptotickej stability a exponenciálnej asymptotickej stability riešení skúmaného systému.

РЕЗЮМЕ

СУЩЕСТВОВАНИЕ И УСТОЙЧИВОСТЬ РЕШЕНИЙ ФУНКЦИОНАЛЬНО-ДИФФЕРЕНЦИАЛЬНОЙ СИСТЕМЫ

Владислав Роса, Братислава

В статье изучается функционально-дифференциальная система, возникшая возмущением какого-то нелинейного дифференциального уравнения. Доказываются теоремы существования решения для этой системы (также максимального и минимального решений) и определены достаточные условия для устойчивости, асимптотической устойчивости и экспоненциально-асимптотической устойчивости решений исследуемой системы.