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CONSTRUCTION OF MEASURE FROM A CONTENT

DANUTA UČNÍKOVÁ, Bratislava

This paper contains several remarks on the article "Construction of measure from a content" by J. Kalas (see [1]).

The construction given in the present paper is a slight modification of that used by J. Kalas in [1] but it yields the same measure. It turns out, however, that some assumptions from [1] can be omitted either because they are implied by the remaining ones or because the proof can be modified so that they become superfluous.

The proofs which can be found in [1] are not mentioned here. The notation from [1] is kept in this work except for the notation for axioms (here denoted by (i)—(x) and (1)—(3)). The notation $o_\sigma(\mathscr{X})$ is used for a σ-ring generated by the system $\mathscr{X}$.

Let us first outline in brief the method used by J. Kalas.

Let $\mathscr{U}$, $\mathscr{B}$ be systems of subsets of a set X. Let the pair $(\mathscr{U}, \mathscr{B})$ fulfil the following assumptions:

(i) $\emptyset \in \mathscr{U} \cap \mathscr{B}$

(ii) if $A_1, A_2 \in \mathscr{U}$, then $A_1 \cup A_2 \in \mathscr{U}$

(iii) if $B_n \in \mathscr{B}$ $(n = 1, 2, \ldots)$, then $\bigcup_{n=1}^{\infty} B_n \in \mathscr{B}$

(iv) if $A \in \mathscr{U}$; $B_1, B_2 \in \mathscr{B}$; $A \subset B_1 \cup B_2$, then there exist such sets $A_1, A_2 \in \mathscr{U}$ that $A_1 \subset B_1$, $A_2 \subset B_2$, $A = A_1 \cup A_2$

(v) if $A \in \mathscr{U}$, $B_n \in \mathscr{B}$ $(n = 1, 2, \ldots)$, $A \subset \bigcup_{n=1}^{\infty} B_n$, then there exists $m \in N$ (natural) such that $A \subset \bigcup_{n=1}^{m} B_n$

(vi) if $A \in \mathscr{U}$, $B \in \mathscr{B}$, then $A - B \in \mathscr{U}$

(vii) if $A \in \mathscr{U}$, $B \in \mathscr{B}$, then $B - A \in \mathscr{B}$

(viii) if $A \in \mathscr{U}$, then there exist sets $B \in \mathscr{B}$, $C \in \mathscr{U}$ such that $A \subset B \subset C$

(ix) if $A_1, A_2 \in \mathscr{U}$, then $A_1 \cap A_2 \in \mathscr{U}$

(x) if $B_1, B_2 \in \mathscr{B}$, then $B_1 \cap B_2 \in \mathscr{B}$

Let λ be a content on the system $\mathscr{U}$ with the following properties:

(1) if $A \in \mathscr{U}$, then $\lambda(A) = \inf\{\lambda(C): A \subset B \subset C,\ B \in \mathscr{B},\ C \in \mathscr{U}\}$

(2) if $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint sets from $\mathscr{U}$, then for every $\varepsilon > 0$ there exists $m \in N$ such that for every $k \in N$ it is true that $\lambda\left(\bigcup_{n=m+1}^{m+k} A_n\right) < \varepsilon$

(3) if $\{A_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of sets from $\mathscr{U}$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$, then $\lim_{n \to \infty} \lambda(A_n) = 0$

Theorem 1. The set function $\bar{\lambda}$ defined for $B \in \mathscr{B}$ by the relation $\bar{\lambda}(B) = \sup\{\lambda(A): B \supset A \in \mathscr{U}\}$ has the following properties:

1. $\bar{\lambda}(\emptyset) = 0$
2. $\bar{\lambda}(A) = \lambda(A)$ for every $A \in \mathscr{U} \cap \mathscr{B}$
3. $\bar{\lambda}$ is nonincreasing on $\mathscr{B}$
4. $\bar{\lambda}$ is σ-subadditive on $\mathscr{B}$
5. $\bar{\lambda}$ is σ-additive on $\mathscr{B}$

Theorem 2. Let P_2 be a system of those sets $E \subset X$ for which it is true that $\inf\{\bar{\lambda}(B - A): A \subset E \subset B,\ A \in \mathscr{U},\ B \in \mathscr{B}\} = 0$. Then P_2 is a σ-ring.

Theorem 3. The assertion $o_\sigma(\mathscr{U} \cup \mathscr{B}) \subset P_2$ is true.

Theorem 4. For $E \in P_2$ let us define a set function $\mu(E) = \sup\{\lambda(A): A \subset E,\ A \in \mathscr{U}\}$. Then the following assertions are true:

1. $\mu / \mathscr{U} = \lambda$
2. $\mu / \mathscr{B} = \bar{\lambda}$
3. μ is nonincreasing on P_2
4. for every $E \in P_2$ is $\mu(E) = \inf\{\bar{\lambda}(B): E \subset B,\ B \in \mathscr{B}\}$
5. μ is additive on P_2

Theorem 5. A set function μ is upper semi-continuous at $\emptyset$.

Remark. According to Theorems 4 and 5, μ is a nonnegative additive set function defined on a σ-ring and upper semi-continuous at $\emptyset$, thus μ is a measure.

Theorem 6. The measure μ is a finite and complete (i.e. if $E \in P_2$ and $\mu(E) = 0$, then $F \subset E$ implies $F \in P_2$) set function on σ-ring P_2.

Remark. Usually we consider the restriction of the measure μ on the system $o_\sigma(\mathscr{U} \cup \mathscr{B})$.

As much about the construction itself. And now some remarks.

The assumptions (i)—(x) and (1)—(3) are rather strong. Thus the natural question is if some of them cannot be omitted.

Theorem 7. Let the pair $(\mathscr{U}, \mathscr{B})$ fulfil the assumptions (v), (vii) and (viii). If

$\{A_n\}_{n=1}^{\infty}$ is a nonincreasing sequence, where A_n $(n=1, 2, \ldots)$ are nonempty sets from $\mathscr{U}$, then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

Proof. According to (viii), for A_1 there exists $B \in \mathscr{B}$ such that $B \supset A_1 \supset A_2 \supset \ldots$ Let $B_n = B - A_n$ $(n=1, 2, \ldots)$. According to (vii), $B_n \in \mathscr{B}$ $(n=1, 2, \ldots)$. Let $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Then $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} (B - A_n) = B - \bigcap_{n=1}^{\infty} A_n = B \supset A_1$ and by (v) there exists $m \in N$, for which it is true that $A_m \subset A_1 \subset \bigcup_{n=1}^{m} B_n = \bigcup_{n=1}^{m} (B - A_n) = B - A_m$. We have $A_m \subset B - A_m$, thus $A_m = \emptyset$.

Remark. Let λ be a content on $\mathscr{U}$. According to the previous theorem, it is evident that the property (3) follows from (v), (vii), and (viii), therefore it is useless to introduce it as an axiom.

Theorem 8. Let the pair $(\mathscr{U}, \mathscr{B})$ fulfil the axioms (vi)—(viii). Then for every $A_1, A_2 \in \mathscr{U}$ we have $A_1 \cap A_2 \in \mathscr{U}$.

Proof. If $A_1 \in \mathscr{U}$, then according to (viii) there exist $B_1 \in \mathscr{B}$ and $C_1 \in \mathscr{U}$ such that $A_1 \subset B_1 \subset C_1$. Since $A_1 \cap A_2 = A_1 - (B_1 - A_2)$, and (by (vii)) $B_1 - A_2 \in \mathscr{B}$, we have (according to (vi)) $A_1 \cap A_2 = A_1 - (B_1 - A_2) \in \mathscr{U}$.

Remark. It is obvious that the axiom (ix) also can be omitted, because it follows from (vi)—(viii). Let us consider the axiom (x), which is somehow dual to (ix).

Example. Let $X = \{1, \ldots, 6\}$, $\mathscr{U} = \{\emptyset, \{1, 3\}, \{2, 3\}, \{3\}, \{1, 2, 3\}\}$, $\mathscr{B} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{4, 5\}, \{5, 6\}, \{4, 5, 6\}, \{1, 2, 3\}, \{1, 4, 5\}, \{1, 5, 6\}, \{2, 4, 5\}, \{2, 5, 6\}, \{1, 4, 5, 6\}, \{2, 4, 5, 6\}, \{1, 2, 4, 5\}, \{1, 2, 5, 6\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 5, 6\}, \{1, 2, 4, 5, 6\}, X\}$. Then the pair $(\mathscr{U}, \mathscr{B})$ fulfils the assumptions (i)—(viii). For $A \in \mathscr{U}$ we set $\lambda(A) = 1$. Then λ is a content on $\mathscr{U}$ with the properties (1) and (2). But the assumption (x) is not fulfilled since, for example, $\{4, 5\} \cap \{5, 6\} \notin \mathscr{B}$.

Remark. The previous example shows that (x) need not be fulfilled even if X is a finite set. But the mentioned construction of measure from a content can be used also in that case. Namely the axiom (x) is necessary only to prove Theorem 5. But there is another way to prove that μ is a measure on P_2.

Theorem 9. If $(\mathscr{U}, \mathscr{B})$ satisfies the assumptions (i)—(viii) and λ is a content on $\mathscr{U}$ with the properties (1) and (2), then the set function μ defined in Theorem 4 is σ-additive on P_2.

Proof. From Theorems 4 and 6 we know that μ is a nonincreasing finite additive function and for $E \in P_2$ it is true that $\mu(E) = \inf\{\bar{\lambda}(B) : E \subset B, B \in \mathscr{B}\}$. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint sets from P_2. Then $\sum_{n=1}^{m} \mu(E_n) = \mu\left(\bigcup_{n=1}^{m} E_n\right) \leq \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$ and for $m \to \infty$ we have $\sum_{r=1}^{\infty} \mu(E_n) \leq \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$.

Now the reverse nonequality. Since $E_n \in P_2$ $(n=1, 2, \ldots)$, for an arbitrary $\varepsilon > 0$ there exist $B_n \in \mathscr{B}$ such that $E_n \subset B_n$ and $\mu(E_n) > \bar{\lambda}(B_n) - \frac{\varepsilon}{2^n}$ $(n=1, 2, \ldots)$.

Thus $$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \bar{\lambda}\left(\bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} \bar{\lambda}(B_n) < \sum_{n=1}^{\infty} \left(\mu(E_n) + \frac{\varepsilon}{2^n}\right) =$$
$$\sum_{n=1}^{\infty} \mu(E_n) + \varepsilon.$$

Remark. As μ is a nonnegative σ-additive set function on the σ-ring P_2, it is a measure.

Remark. The axioms (viii), (1) and (2) guarantee that the measure μ is finite and regular (we mean generalized regularity, i.e. $\mu(E) = \sup\{\mu(A): A \subset E, A \in \mathscr{U}\} = \inf\{\mu(B): E \subset B, B \in \mathscr{B}\}$ for every $E \in P_2$) and on the system $\mathscr{U}$ it coincides with the content λ, from which μ was constructed.

There exist other ways of construction measure from a content which are more general, but the resulting measure does not have the above good properties. Such a generalized method can be found in [2]. The author considers only the axioms (i)—(vii). If axioms (i)—(viii), (1) and (2) are fulfilled, the measure constructed using the method from [2] on the system $o_\sigma(\mathscr{U} \cup \mathscr{B})$ coincides with that constructed according to [1].

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Author's address:
Danuta Učníková
Katedra matematickej analýzy MFFUK,
Mlynská dolina,
842 15 Bratislava

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SÚHRN

KONŠTRUKCIA MIERY Z OBJEMU

Danuta Učníková, Bratislava

V práci sa zovšeobecňuje metóda konštrukcie miery z objemu uvedená v [1].

РЕЗЮМЕ

ПОСТРОЕНИЕ МЕРЫ ИЗ ОБЪЕМА

Данута Учникова, Братислава

В роботе обобщается метод построения меры из объема предложенный Й. Каласом в [1].