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SOME QUESTIONS OF QUASICONTINUITY RELATED TO MATHEMATICAL PROGRAMMING

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Introduction

A study of multifunctions is motivated by numerous applications in different fields. In [1] the idea of multifunctions is exploited in mathematical programming. There are some continuity properties of supremal value function and of the solution set investigated in the paper. This two concepts arise naturally in the development of solution techniques for decomposable mathematical programming problems. The multifunctions determined by inequalities are also studied and some conditions for them to be closed and open are given there. These are properties similar to upper and lower semicontinuity respectively. In this paper we deal with the same problems but for quasicontinuity.

In all what follows X, Y will denote topological spaces and by $\hat{R}$ we denote $\langle -\infty, +\infty \rangle$ with the usual topology.

1. Supremal value function

The supremal value function is termed the function $v\colon X \to \hat{R}$ defined by

$$v(x) = \sup\{f(x, y) : y \in F(x)\},$$

where $f\colon X \times Y \to \hat{R}$ is a function and $F\colon X \to$ is a multifunction. We shall use the capital letters for multifunctions and we shall write $F\colon X \to Y$ instead of $F\colon X \to 2^Y$ for a multifunction whose values are subsets of Y. The case $F(x) = \emptyset$ is also possible. Before studying the properties of supremal value function we introduce some definitions which we shall use.

The notion of quasicontinuity was introduced by Kempisty as follows.

Definition 1. A function $f\colon X \to Y$ is said to be quasicontinuous at a point

$x_0 \in X$ if for any open U containing x_0 and any open V containing $f(x_0)$ there exists a nonempty open set $G \subset U$ such that $f(G) \subset V$. The function is said to be quasicontinuous if it is quasicontinuous at any $x \in X$.

Following definitions are for instance in [2].

Definition 2. A multifunction $F: X \to Y$ is said to be upper (lower) semi-continuous at a point $x_0 \in X$ if for any open V such that $F(x_0) \subset V(F(x_0) \cap V \neq \emptyset)$ there exists an open set U containing x_0 such that $F(x) \subset V(F(x) \cap V \neq \emptyset)$ for any $x \in U$. F is said to be upper (lower) semi-continuous if it is upper (lower) semi-continuous at any $x \in X$.

Definition 3. A multifunction $F: X \to Y$ is said to be upper (lower) semi-quasicontinuous at a point $x_0 \in X$ if for any open U containing x_0 and any open V such that $F(x_0) \subset V(F(x_0) \cap V \neq \emptyset)$ there exists a nonempty open set $G \subset U$ such that $F(x) \subset V$ $(F(x) \cap V \neq \emptyset)$ for any $x \in G$. F is said to be upper (lower) semiquasicontinuous if it is upper (lower) semi-quasicontinuous at any $x \in X$.

We shall denote by u.s.c., l.s.c., u.q.c., l.w.c. the upper semi-continuity, the lower semi-continuity, the upper semi-quasicontinuity, and the lower semi-quasicontinuity respectively.

Following Definition 1. we can introduce the semi-quasicontinuity derived from the semi-continuity[1]) of real valued functions. In this case we shall speak about order semi-continuities and order semi-quasicontinuities to distinguish these from the notions defined above.

Definition 4. If $f(x_0)$ is finite, then a function $f: X \to \hat{R}$ is said to be order upper (lower) semi-quasicontinuous at a point $x_0 \in X$ if for any open U containing x_0 and any $\varepsilon > 0$ there exists a nonempty open set $G \subset U$ such that $f(x) < f(x_0) + \varepsilon (f(x) > f(x_0) - \varepsilon)$ for any $x \in G$. In the case $f(x_0) = +\infty$ a function f is always order upper semi-quasicontinuous at x_0 and it is said to be order lower semi-quasicontinuous at x_0 if the inequality $f(x) > \varepsilon$ holds instead of $f(x) > f(x_0) - \varepsilon$ for any $x \in G$. In the case $f(x_0) = -\infty$ a function f is always order lower semi-quasicontinuous at a point x_0 and it is said to be order upper semi-quasicontinuous at x_0 if the inequality $f(x) < -\varepsilon$ holds instead of $f(x) < f(x_0) + \varepsilon$ for any $x \in G$. The function is said to be order upper (lower) semi-quasicontinuous if it is order upper (lower) semi-quasicontinuous at any $x \in X$.

We shall denote by o.u.s.c., o.l.s.c., o.u.q.c., o.l.q.c. the order upper semi-continuity, the order lower semi-continuity, the order upper semi-quasicontinuity, and the order lower semi-quasicontinuity respectively.

The next definition can be found in [1].

[1]) The definition of the semi-continuity can be found for instance in C. Berge: Topological Spaces. Macmilan. New York, 1963.

Definition 5. A multifunction $F: X \to Y$ is said to be uniformly compact near a point $x_0 \in X$ if there is a neighbourhood U of x_0 such that the closure of the set $F(U) = \bigcup_{x \in U} F(x)$ is compact in Y.

In all theorems following in the first two sections we do not exclude the possibilities that f and v are infinite but for shortness the proofs are given only for f and v finite. The cases when f and v are infinite can be proved with a slight modification of the proofs given here.

Theorem 1. Let F be u.q.c. at a point $x_0 \in X$, let f be o.u.s.c. on $x_0 \times F(x_0)$ and let $F(x_0)$ be compact. Then v is o.u.q.c. at x_0.

Proof. Let U be an open set containing x_0 and let $\varepsilon > 0$. From the supremum property we have

$$f(x_0, y) < v(x_0) + \frac{\varepsilon}{3}$$

for any $y \in F(x_0)$.

Since f is o.u.s.c., for any $y \in F(x_0)$ there exist U_y and V_y neighbourhoods of x_0 and y respectively such that for any (x, z) from $U_y \times V_y$ we have

$$f(x, z) < f(x_0, y) + \frac{\varepsilon}{3} < v(x_0) + \frac{2\varepsilon}{3}. \tag{1}$$

The set $V = \bigcup_{y \in F(x_0)} V_y$ covers $F(x_0)$ and the compactness of $F(x_0)$ implies that there exists the finite subcover $V_{y_1}, \ldots, V_{y_n}$. Denote

$$\tilde{V} = V_{y_1} \cup \ldots \cup V_{y_n} \quad \text{and} \quad \tilde{U} = U_{y_1} \cap \ldots \cap U_{y_n} \cap U$$

where U_{y_i} is the neighbourhood of x corresponding to V_{y_i} for $1 \leq i \leq n$.

Then u.q.c. of F implies that there exists a nonempty open set $G \subset \tilde{U}$ such that $F(x) \subset V$ for any $x \in G$. If now $y \in F(x)$ where $x \in G$, then $y \in V_{y_i}$ for some i, $1 \leq i \leq n$. Because $x \in G$ implies $x \in U_{y_i}$, according to (1) we have

$$f(x, y) < f(x_0, y_i) + \frac{\varepsilon}{3} < v(x_0) + \frac{2\varepsilon}{3}$$

for any $y \in F(x)$ where $x \in G$.

Hence

$$v(x) \leq v(x_0) + \frac{2\varepsilon}{3} < v(x_0) + \varepsilon$$

for any $x \in G$. The o.u.q.c. of v at x_0 is proved.

Theorem 2. Let F be l.q.c. at a point $x_0 \in X$ and let f be o.l.s.c. on $x_0 \times F(x_0)$. Then v is o.l.q.c. at x_0.

Proof. Let U be an open set containing x_0 and let $\varepsilon > 0$. The property of supremum implies that there exists a point $y_0 \in F(x_0)$ such that

$$f(x_0, y_0) > v(x_0) - \frac{\varepsilon}{2}.$$

From the o.l.q.c. of f on $x_0 \times F(x_0)$ it follows that there are U_0, V_0 neighbourhoods of x_0 and y_0 respectively such that

$$f(x, y) > f(x_0, y_0) - \frac{\varepsilon}{2} > v(x_0) - \varepsilon \tag{2}$$

for any $(x, y) \in U_0 \times V_0$.

Let $U_1 = U_0 \cap U$. From the l.q.c. of F it follows that there exists a nonempty open set $G \subset U_1$ such that $F(x) \cap V_0 \neq \emptyset$ for any $x \in G$. It means that for any $x \in G$ there is a point y_x such that $y_x \in F(x) \cap V_0$. Hence a point (x, y_x) belongs to the set $U_0 \times V_0$ and according to (2) we have

$$v(x) \geq f(x, y_x) > v(x_0) - \varepsilon$$

for any $x \in G$. The o.l.q.c. of v at x_0 is proved.

If $Y = \hat{R}$ and f: $X \to Y$, the quasicontinuity of f is equivalent to simultaneous o.u.q.c. and o.l.q.c. provided that the sets G which can be different for individual types of semi-quasicontinuities are not disjoint. Then we can introduce the quasicontinuity for multifunctions as follows.

Definition 6. A multifunction F: $X \to Y$ is said to be quasicontinuous at a point $x_0 \in X$ if for any open set U containing x_0 and any open V_1, V_2 such that $F(x_0) \subset V_1$ and $F(x_0) \cap V_2 \neq \emptyset$ there exists a nonempty open set $G \subset U$ such that $F(x) \subset V_1$ and $F(x) \cap V_2 \neq \emptyset$ for any $x \in G$.

The next theorem is given without a proof because it is a combination of the previous theorems' proofs.

Theorem 3. Let F be quasicontinuous at a point $x_0 \in X$, let f be continuous on $x_0 \times F(x_0)$ and let $F(x_0)$ be compact. Then v is quasicontinuous at x_0.

2. Solution set

When v is the supremal value function, then the multifunction M: $X \to Y$ defined by

$$M(x) = \{y \in F(x) : f(x, y) = v(x)\}$$

describes its solution set — the set of the solution points.

Before formulating the theorems concerning the quasicontinuity of this multifunction we give some lemmas in order to make the proof of the main theorem more intelligible. The first lemma can be found in [2].

Lemma 1. Let X be a first countable Hausdorff topological space. Let x_0 be a point which is not isolated. Suppose that $\{P_k\}_{k=1}^{\infty}$ is a sequence of properties such that to any neighbourhood U of x_0 a sequence $\{M_k\}_{k=1}^{\infty}$, $M_k \subset U$ of nonempty open sets exists such that P_k is satisfied on M_k. Then a descending base $\{W_k\}_{k=1}^{\infty}$ of neighbourhoods of x_0 and a sequence $\{G_k\}_{k=1}^{\infty}$ of mutually disjoint open sets exist such that $G_k \subset W_k$, $G_k \cap W_{k+1} = \emptyset$, P_k is satisfied on G_k for $k = 1, 2, \ldots$. Evidently the set $A = \bigcup_{k=1} G_k \cup \{x_0\}$ is quasiopen.

Remark 1. The sequence $\{W_k \cap A\}_{k=1}^{\infty}$ is a base of neighbourhoods of x_0 in a topology induced on A and if the sequence $\{P_k\}_{k=1}^{\infty}$ satisfies $P_{k+1} \Rightarrow P_k$ for any integer k, then the property P_k is satisfied on $W_k \cap A$. A set A is said to be quasiopen if $A \subset \overline{A^0}$.[2])

Lemma 2. Let X be a first countable Hausdorff topological space and let Y be a metric space. If F is quasicontinuous at a point $x_0 \in X$, f is continuous on $x_0 \times F(x_0)$ and if $F(x_0)$ is compact, then there exists a quasiopen set A containing x_0 such that F/A is u.s.c. at x_0 and v/A is continuous at x_0.

Proof. If x_0 is isolated then it is sufficient to take $A = \{x_0\}$ and the theorem is proved.

Suppose x_0 is not isolated. Let U be an open set containing x_0, let V be an open set such that $F(x_0) \subset V$ and let $\varepsilon > 0$. From the property of supremum it follows that there exists $y_0 \in F(x_0)$ such that

$$f(x_0, y_0) > v(x_0) - \frac{\varepsilon}{2}$$

and it also follows that

$$f(x_0, y) < v(x_0) + \frac{\varepsilon}{3}$$

for any $y \in F(x_0)$.

From the continuity of f for any $y \in F(x_0)$ there exists U_y, V_y neighbourhoods of x_0 and y respectively such that for any (x, z) from $U_y \times V_y$ we have

$$f(x, z) < f(x_0, y) + \frac{\varepsilon}{3} < v(x_0) + \frac{2\varepsilon}{3} \tag{3}$$

and U_{y_0}, V_{y_0} can be found such that moreover we have

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[2]) The symbols $\bar{A}$, A°, A' are used to denote the closure, the interior and the complement of A respectively.

$$f(x, z) > f(x_0, y_0) - \frac{\varepsilon}{2} > v(x_0) - \varepsilon \tag{4}$$

for any $(x, y) \in U_{y_0} \times V_{y_0}$.

The neighbourhoods V_y can be chosen such that $V_y \subset V$. If $\tilde{V} = \bigcup_{y \in F(x_0)} V_y$ then $F(x_0) \subset \tilde{V} \subset V$. The compactness of $F(x_0)$ implies that the sets $V_{y_1}, \ldots, V_{y_n}$ exist such that when we denote $\tilde{\tilde{V}} = \bigcup_{i=1}^{n} V_{y_i}$ then $F(x_0) \subset \tilde{\tilde{V}} \subset V$. Denote

$$\tilde{U} = U_{y_1} \cap \ldots \cap U_{y_n} \cap U_{y_0} \cap U$$

where U_{y_i} is the neighbourhood of x_0 corresponding to V_{y_i} for $1 \leq i \leq n$.

The quasicontinuity of F implies that there exists a nonempty open set $G \subset \tilde{U}$ such that $F(x) \subset \tilde{\tilde{V}} \subset V$ and $F(x) \cap V_{y_0} \neq \emptyset$ for any $x \in G$. Then the following two facts are obvious for any $x \in G$. There exists $y_x \in F(x) \cap V_{y_0}$ and if $y \in F(x)$ then $y \in V_{y_i}$ for some i, $1 \leq i \leq n$. Hence $(x, y_x) \in U_{y_0} \times V_{y_0}$ and $(x, y) \in U_{y_i} \times V_{y_i}$. From the supremum property and according to (3) and (4) we have

$$v(x) \geq f(x, y_x) > v(x_0) - \varepsilon \tag{5}$$

and

$$v(x) \leq v(x_0) + \frac{2\varepsilon}{3} < v(x_0) + \varepsilon \tag{6}$$

for any $x \in G$.

Combining inequalities (5) and (6) we have $|v(x) - v(x_0)| < \varepsilon$ for any $x \in G$. We have also proved that $F(x) \subset V$ for any $x \in G$.

Since Y is metric and $F(x_0)$ is compact, there exists a descending sequence $\{V_k\}_{k=1}^{\infty}$ of open sets such that $F(x_0) \subset V_k$ and for any open set W there exists k such that $V_k \subset W$. If now the property P_k is formulated as $|v(x) - v(x_0)| < \frac{1}{k}$ and $F(x) \subset V_k$, then the assumptions of previous lemma are satisfied and the existence of a quasiopen set A with properties formulated in Lemma 1 is guaranteed. Since $P_{k+1} \Rightarrow P_k$, there exists a descending base $\{W_k\}_{k=1}^{\infty}$ of neighbourhoods of x_0 such that P_k is satisfied on $W_k \cap A$. This proves that F/A is u.s.c. at x_0 and v/A is continuous at x_0.

Since the proof of the following lemma does not essentially differ from the proof of Theorem 8 in [1], it will be omitted.

Lemma 3. Let X, Y be first countable topological spaces, let X be Hausdorff and Y be regular. Let F be uniformly compact near x_0 and let $F(x_0)$ be closed. If there exists a quasiopen set A containing x_0 such that F/A is i.s.c. at x_0 and v/A is continuous at x_0, then M/A is u.s.c. at x_0.

The last lemma that we shall need is formulated as Theorem 3 in [3].

Lemma 4. Let X be a first countable Hausdorff space, let Y be a pseudometric space and let $F(x_0)$ be compact. Then F is u.q.c. at a point $x_0 \in X$ if and only if there exists a quasiopen set A containing x_0 such that F/A is u.s.c. at x_0.

Theorem 4. Let X be a first countable Hausdorff topological space and let Y be a metric space. If F is quasicontinuous at a point $x_0 \in X$ and uniformly compact near x_0, if $F(x_0)$ is closed and if f is continuous on $x_0 \times F(x_0)$, then M is u.q.c. at x_0.

Proof. Since the assumptions of Lemma 2 are satisfied, there exists a quasiopen set containing x_0 such that F/A is u.s.c. at x_0 and v/A is continuous at x_0. (The compactness of $F(x_0)$ follows from its closedness and from the uniformly compactness of F near x_0.) The assumptions of Lemma 3 are satisfied because Y is metric and so M/A is u.s.c. at x_0. Then according to Lemma 4, the assumptions of which are also satisfied, M is u.q.c. at x_0. The theorem is proved.

Corollary 4.1. Let X, Y, F and f fulfil the same conditions as in Theorem 4. If M is nonempty in some neighbourhood of x_0 and if $M(x_0)$ is single-valued, then M is quasicontinuous at x_0.

Proof. The quasicontinuity of M at x_0 follows immediately from the u.q.c. of M at x_0 and from the facts that $M(x_0)$ is singlevalued and M is nonempty in some neighbourhood of x_0.

Remark 2. In the case when we operate with ε-optimal solutions for the evaluation of $v(x)$ rather than with the exact optimal solutions defined by $M(x)$, we can work with this modification of M

$$M^*(x, \varepsilon) = \{y \in F(x) : v(x) \leq f(x, y) + \varepsilon\} .$$

In a similar way as we have proved the u.q.c. of multifunction M we can prove that under the same conditions on X, Y, F and f as in Theorem 4, a multifunction M^* is also u.q.c. on $x_0 \times R^+$. We can even obtain a stronger result which states that $M^*/A \times R^+$ is u.s.c. on $x_0 \times R^+$, where A is a quasiopen set containing x_0.

3. Multifunctions occurring in nonlinear programming

Some sets in mathematical programming are often determined by a system of inequalities. Usually the set of feasible points in a nonlinear programming is described by such a system. We shall therefore study some properties of the multifunctions determined by a system of inequalities.

For instance the canonical convex programming problem can be stated as $\sup\{\hat{f}(y) : \hat{g}(y) \leq 0, y \in Y\}$ [3]), where $\hat{g}: Y \to \hat{R}^m$ and $\hat{f}: Y \to \hat{R}$ are such that $-\hat{f}$ and

[3]) If $x, y \in \hat{R}^m$, then the statement $x \leq y (x < y)$ will mean that $x_i \leq y_i (x_i < y_i)$ for each pair of corresponding components x_i, y_i.

$\hat{g}$ are convex functions on the convex set Y. The perturbation function for this problem is
vex set Y. The perturbation function for this problem is

$$v(x)=\sup\{\hat{f}(y):\hat{g}(y)\leq x,\ y\in Y\}\ .$$

Let now $f(x, y)=\hat{f}(y)$ and $g(x, y)=\hat{g}(y)-x$, where f and g are from $X\times Y$ into $\hat{R}^m$. If we denote $F(x)=\{y\in Y:\ g(x, y)\leq 0\}$ we can see that the perturbation function is of the form

$$v(x)=\sup\{f(x, y):\ y\in F(x)\}\ .$$

Thus we obtained the supremal value function properties of which we have studied in the first section. Since the quasicontinuity of v depends on the quasicontinuity of F, it is natural to find out under what conditions the multifunction F determined by inequalities is quasicontinuous.

A sufficient condition for F to be u.q.c. provides the following theorem.

Theorem 5. Let A be a quasiopen set containing a point $x_0\in X$ and let $g/A\times Y$ be o.l.s.c. on $x_0\times Y$. If F is uniformly compact near x_0, then F is u.q.c. at x_0.

Proof. Suppose that the assumptions are satisfied and F is not u.q.c. at x_0. Then there exist open sets U_0, V_0 such that $x_0\in U_0$, $F(x_0)\subset V_0$ and for any nonempty open set $G\subset U_0$ two points $x_G\in G$ and $y_G\in F(x_G)\cap V_0'$ exist. Since F is uniformly compact at x_0, there exists a neighbourhood $\tilde{U}$ of x_0 such that $\overline{F(\tilde{U})}$ is compact.

Consider now the system of sets

$$G_U=U\cap\tilde{U}\cap U_0\cap A^0\ ,$$

where U runs over the system $\mathcal{U}$ of all neighbourhoods of x_0. Then there exist two nets $\{x_U\}_{U\in\mathcal{U}}$ and $\{y_U\}_{U\in\mathcal{U}}$ such that $x_U\in G_U$ and $y_U\in F(x_U)\cap V_0'$. Evidently $x_U\to x_0$ and because all x_U are contained in $\tilde{U}$ there exists a point y_0 such that $y_{U_\gamma}\to y_0$, where $\{y_{U_\gamma}\}_{\gamma\in\Gamma}$ is some subnet of the net $\{y_U\}_{U\in\mathcal{U}}$.

From the o.l.s.c. of $g/A\times Y$ at a point (x_0, y_0) we have

$$g(x_0, y_0)\leq\liminf g(x_{U_\gamma}, y_{U_\gamma})\leq 0\ .$$

Hence we have $y_0\in F(x_0)$ and at the same time since $y_{U_\gamma}\in V_0'$, we get $y_0\in V_0'$. This contradicts $F(x_0)\subset V_0$ and the u.q.c. of F at x_0 is proved.

Denoting $J=\{x\in X:\ g(x, y)\leq 0$ for some $y\in Y\}$ we can state a necessary condition for F to be l.q.c.. The simple proof is omitted.

Theorem 6. If F is l.q.c. at a point $x_0\in J$, then $x_0\in\overline{J^0}$.

The Definition 4 can be naturally extended to functions with values in $\hat{R}^m$. If $g(x_0)$ is finite then a function $g: X\to\hat{R}^m$ can be said to be order upper semi-quasicontinuous at a point $x_0 \in X$ if for any open set U containing x_0 and any $\varepsilon > 0$ there exists a nonempty open set $G \subset U$ such thatk $g_i(x) < g_i(x_0) + \varepsilon$[4]) for any $x \in G$ and any $i = 1, 2, \ldots, m$. The cases when $g(x_0)$ is infinite can be reformulated similarly. Using this definition of o.u.q.c. we can give a sufficient condition for F to be l.q.c..

Theorem 7. Let Y be a convex and normed space. Let the sections g^y be o.u.q.c. at a point $x_0 \in X$ for any $y \in F(x_0)$ and let all components of all sections g_x be convex. If there exists $y_0 \in Y$ such that $g(x_0, y_0) < 0$, then F is l.q.c. at x_0.

Proof. If we consider $g_{\max}(x, y) = \max\{g_1(x, y), \ldots, g_m(x, y)\}$ instead of g, the convexity and the quasicontinuity as it was defined above are preserved. It is also evident that

$$F_{g_{\max}}(x) = \{y \in Y : g_{\max}(x, y) \leqslant 0\} = \\ = \{y \in Y : g_i(x, y) \leqslant 0 \text{ for } i = 1, \ldots, m\} = \\ = F(x)$$

and so we can assume without loss of generality that $g: X \times Y \to \hat{R}$.

Let now U be a neighbourhood of x_0, V be an open set such that $F(x_0) \cap V \neq \emptyset$ and let $\hat{y} \in F(x_0) \cap V$. Suppose that for any point $y \in F(x_0) \cap V$ the inequality $g(x_0, y) \geqslant 0$ holds. Then $y_0 \notin V$ and there exists $\lambda \in (0, 1)$ such that $y_\lambda = (1 - \lambda)\hat{y} + \lambda y_0$ belongs to V. Here we assume that V is absorbing.

From the convexity of g_x we have

$$g(x_0, y_\lambda) \leqslant (1 - \lambda) g(x_0, \hat{y}) + \lambda g(x_0, y_0) = \lambda g(x_0, y_0) < 0 .$$

This contradicts the assumptions and that is why we can suppose that in $F(x_0) \cap V$ some point $\bar{y}$ exists such that $g(x_0, \bar{y}) < 0$.

Since $g^{\bar{y}}$ is o.u.q.c. at x_0, there exists a nonempty open set $G \subset U$ such that $g(x, \bar{y}) < 0$ for any $x \in G$. Hence $\bar{y} \in F(x) \cap V$ for any $x \in G$ and F is l.q.c. at x_0.

If we denote $I(x) = \{y \in Y: g(x, y) < 0\}$ we can substitute convexity by a more general condition.

Theorem 8. If the sections g^y are o.u.q.c. at a point $x_0 \in X$ for any $y \in I(x_0)$ and if $F(x_0) \subset \overline{I(x_0)}$, then F is l.q.c. at x_0.

Proof. Following the same consideration as in the previous proof, we can consider g with values in $\hat{R}$ instead of $\hat{R}^m$.

Let U be a neighbourhood of x_0 and let V be an open set such that $F(x_0) \cap V \neq \emptyset$. Then $\overline{I(x_0)} \cap V \neq \emptyset$ and there exists $\hat{y} \in \overline{I(x_0)} \cap V$. Hence also a point $\bar{y}$ exists such that $\bar{y} \in I(x_0) \cap V$ and so $g(x_0, \bar{y}) < 0$.

From the o.u.q.c. of $g^{\bar{y}}$ at x_0 it follows that a nonempty open set $G \subset U$ exists

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[4]) g_i denotes the component of g. It means $g(x) = (g_1(x), \ldots, g_m(x))$.

such that $g(x, \bar{y})<0$ for any $x \in G$. We can see that $\bar{y} \in F(x)$ and so $F(x) \cap V \neq \emptyset$ for any $x \in G$. The l.q.c. of F at x_0 is proved.

We give the last sufficient condition for the l.q.c. of F.

Theorem 9. Let X, Y be normed spaces, let Y be convex and bounded and let g be quasiconvex[5]) on $X \times Y$. Let $D \subset J$ be a set such that $d(x_0, D)>0$, where the last symbol denotes a distance between a point and a set. If for any neighbourhood U of x_0 a nonempty open set $G \subset U$ exists such that there are $\hat{x} \in D$ and $\lambda_x \in \langle 0, 1\rangle$ such that $x=\lambda_x \hat{x}+(1-\lambda_x)x_0$ for any $x \in G$, then F is l.q.c. at a point $x_0 \in X$.

Proof. By the assumptions if $x \in G$, where $G \subset U$ is the set corresponding to a neighbourhood U of x_0, then

$$x=\lambda_x \hat{x}+(1-\lambda_x)x_0$$

where $\lambda_x \in \langle 0, 1\rangle$ and $\hat{x} \in D$. Since $D \subset J$, there exists $\hat{y} \in Y$ such that $g(\hat{x}, \hat{y}) \leqslant 0$.

If V is an open set such that $F(x_0) \cap V \neq \emptyset$, then there exists a point $y_0 \in F(x_0) \cap V$. Consider the points

$$y_x=\lambda_x \hat{y}+(1-\lambda_x)y_0 .$$

Since Y is bounded there exists a real k such that $\|y\|<k$ for any $y \in Y$. Then we have

$$\|y_x-\hat{y}_0\|=\lambda_x \|\hat{y}-y_0\| \leqslant 2k\lambda_x .$$

If $\lambda_x<\dfrac{\varepsilon}{2k}=\varepsilon_0$, then $\|y_x-y_0\|<\varepsilon$ and the point y_x belongs to $S(y_0, \varepsilon)$, where $S(y_0, \varepsilon)$ denotes the ε-neighbourhood of y_0. Since we can find ε such that $S(y_0, \varepsilon) \subset V$, we need only to find conditions for $\lambda_x<\varepsilon_0$.

Denote $L=d(x_0, D)>0$. If $\delta=\varepsilon_0 \cdot L$ and $x \in S(x_0, \delta)$, then we have

$$\lambda_x=\frac{\|x-x_0\|}{\|\hat{x}-x_0\|}<\frac{\varepsilon_0 \cdot L}{L}=\varepsilon_0 .$$

Now if $\tilde{U}=S(x_0, \delta) \cap U$ then a nonempty open set $G \subset \tilde{U}$ exists such that any $x \in G$ can be written in the form of

$$x=\lambda_x \hat{x}+(1-\lambda_x)x_0 ,$$

where $\hat{x} \in D$ and $\lambda_x \in \langle 0, 1\rangle$, $\lambda_x<\varepsilon_0$. Hence $y_x \in V$.

From the quasiconvexity of g we have

$$g(x, y_x) \leqslant \max \{g(\hat{x}, \hat{y}), g(x_0, y_0)\} \leqslant 0 .$$

This implies that $y_x \in F(x)$ and so $F(x) \cap V \neq \emptyset$ for any $x \in G$. The proof is finished.

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[5]) A function g is said to be quasiconvex if for any $\lambda \in \langle 0, 1\rangle$ and $(x_1, y_1), (x_2, y_2) \in X \times Y$ we have $g(\lambda \cdot x_1+(1-\lambda) \cdot x_2, \lambda \cdot y_1+(1-\lambda) \cdot y_2) \leqslant \max \{g(x_1, y_1), g(x_2, y_2)\}$.

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SÚHRN

NIEKTORÉ OTÁZKY KVÁZISPOJITOSTI TÝKAJÚCE SA MATEMATICKÉHO PROGRAMOVANIA

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V práci sa skúmajú otázky kvázispojitosti supremovo hodnotovej funkcie a multifunkcie popisujúcej množinu optimálnych riešení. Uvedené sú aj niektoré podmienky, za ktorých je multifunkcia, určená nerovnosťami, polokrázispojitá zhora, resp. zdola.

РЕЗЮМЕ

НЕКОТОРЫЕ ВОПРОСЫ КВАЗИ-НЕПРЕРЫВНОСТИ КАСАЮЩИЕСЯ МАТЕМАТИЧЕСКОГО ПРОГРАММИРОВАНИЯ

Ондрей Натэр, Братислава

В работе исследуются вопросы квази-непрерывности супремно значительной функции и многозначного отображения описывающего множество оптимальных точек. Также даются условия, при которых многозначное отображение, определенное неревенствами, является полуквази-непрерывным сверху или же снизу.