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# ON LIPSCHITZ SELECTIONS OF LIPSCHITZ MULTIFUNCTIONS

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**1. Introduction.** The aim of the present paper is to give some sufficient conditions for the existence of Lipschitz selections of Lipschitz multifunctions. Let $R$ denote the set of reals, let $R^k$ be the $k$-dimensional Euclid space and let $C(R^k)$ denote the family of all convex compact nonvoid subsets of $R^k$. The problem treated in the present paper has its origin in the theory of the generalized differential equations. It can be formulated in the following way: Has every Lipschitz function $F$: $R^m \to C(R^n)$ a Lipschitz selection? ($C(R^n)$ is furnished with the Hausdorff metric (see e.g. [1], [3]); a function $f$: $R^m \to R^n$ is said to be a selection for $F$ if $f(x) \in F(x)$ for each $x \in R^m$). Note that there are known some sufficient conditions for the existence of a Lipschitz selection (see e.g. [2], [4]). These differ from our results or use some different methods in the argumentation.

Recall the form of the Hausdorff metric that we shall use in the next text, giving some of its properties.

**Definition.** Let $(R^n, \varrho)$ be a metric space and let $A, B \in C(R^n)$. Put $d(A, B) = \max\{\sup_{x \in B} \varrho(x, A), \sup_{x \in A} \varrho(x, B)\}$. The function $d$: $C(R^n) \times C(R^n) \to R$ is called the Hausdorff metric.

**Lemma 1.** Let $A$, $B \in C(R^n)$. Then $d(A, B) = \max\left\{\max_{x \in B} \varrho(x, A), \max_{x \in A} \varrho(x, B)\right\}$.

**Lemma 2.** Let $\bar{K}(Z, \varepsilon) = \{t: \varrho(t, Z) \leq \varepsilon\}$. Then $d(A, B) \leq \varepsilon$ if and only if $A \subset \bar{K}(B, \varepsilon)$ and $B \subset \bar{K}(A, \varepsilon)$ (see [3], p. 224).

**2. Results and proofs.** Further we shall suppose that $(R^n, \varrho)$ is a metric space with $\max_{i=1,\dots,n}\{|u_i - v_i|\} \leq \varrho(u, v) \leq \sum_{i=1}^{n} |u_i - v_i|$ for any $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$.

**Theorem 1.** Let $F$: $D \to C(R^n)((D, \delta)$ — a metric space) fulfil a Lipschitz condition of the form $d(F(x), F(y)) \leq L\delta(x, y)$, $x, y \in D$. Let for each $x \in D J(x) = \langle a_1, b_1 \rangle \times \ldots \times \langle a_n, b_n \rangle$ be the minimal $n$-dimensional interval with respect to the inclusion, such that $F(x) \subset J(x)$. If there are numbers $\lambda_1, \ldots, \lambda_n (0 \leq \lambda_i \leq 1)$ such that $c(x) = (\lambda_1 a_1 + (1-\lambda_1) b_1, \ldots, \lambda_n a_n + (1-\lambda_n) b_n) \in F(x)$ for each $x \in D$, then $f\colon D \to R^n$, $f(x) = c(x)$, is a selection for $F$ with the Lipschitz constant $nL$.

**Proof.** Let $x, y \in D$ and put $\varepsilon = L\delta(x, y)$. Then $d(F(x), F(y)) \leq \varepsilon$. The last inequality is equivalent, according to Lemma 2, to inclusions $F(x) \subset \bar{K}(F(y), \varepsilon)$ and $F(y) \subset \bar{K}(F(x), \varepsilon)$. On the contrary it can be easy to check using the above inclusions that if $J(x) = \langle a_1, b_1 \rangle \times \ldots \times \langle a_n, b_n \rangle$ and $J(y) = \langle a'_1, b'_1 \rangle \times \ldots \times \langle a'_n, b'_n \rangle$, then $|a_i - a'_i| \leq \varepsilon$ and $|b_i - b'_i| \leq \varepsilon$ hold for each $i = 1, \ldots, n$. Consequently $\varrho(f(x), f(y)) = \varrho(c(x), c(y)) \leq \sum_{i=1}^{n} (\lambda_i |a_i - a'_i| + (1-\lambda_i) |b_i - b'_i|) \leq n\varepsilon = nL\delta(x, y)$. Q.E.D.

**Corollary 1.** If $F\colon D \to C(R^n)$, $n = 1, 2$, fulfils the Lipschitz condition, then $F$ admits a Lipschitz selection.

If $n = 1$, then the statement of Corollary 1 is obvious. Let $n = 2$ and $J(x) = \langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle$. We show that it is sufficient to put $\lambda_1 = \lambda_2 = \frac{1}{2}$. There are points $A_1, A_2, B_1, B_2$ in $F(x)$ such that $A_1 = (a_1, u_1)$, $u_1 \in \langle a_2, b_2 \rangle$, $A_2 = (u_2, a_2)$, $u_2 \in \langle a_1, b_1 \rangle$, $B_1 = (b_1, v_1)$, $v_1 \in \langle a_2, b_2 \rangle$, $B_2 = (v_2, b_2)$, $v_2 \in \langle a_1, b_1 \rangle$. Since $F(x)$ is a convex set the segments $[A_1, A_2]$ and $[B_1, B_2]$ are contained in $F(x)$ and they are in general situated on opposite sides of the diagonal $w_1 = [(a_1, b_2), (b_1, a_2)]$. Hence in general the diagonal $w_2 = [(a_1, b_1), (a_2, b_2)]$ has common points $A$ and $B$ with the segments $[A_1, A_2]$ and $[B_1, B_2]$ respectively, which are situated on opposite sides of $w_1$. Consequently $c(x) = \left(\frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2)\right) \in [A, B]$ and the inclusion $[A, B] \subset F(x)$ implies $c(x) \in F(x)$. Q.E.D.

**Remark.** The method of the proof of Corollary 1 is not acceptable for $n \geq 3$.

**Theorem 2.** Let $F\colon D \to C(R^n)$ be a Lipschitz function with the Lipschitz constant $L$. Let $F$ have bounded values, i.e. there is $M > 0$ such that $\operatorname{diam}\{F(x)\} \leq M$ holds for each $x \in D$, and let $J(x) = \langle a_1, b_1 \rangle \times \ldots \times \langle a_n, b_n \rangle$ be the interval determined by Theorem 1. If there are Lipschitz functions $\lambda_1(x), \ldots, \lambda_n(x)$ $(\lambda_i\colon D \to \langle 0, 1 \rangle)$ with Lipschitz constants $L_1, \ldots, L_n$ such that $c(x) = (\lambda_1(x) a_1 + (1-\lambda_1(x)) b_1, \ldots, \lambda_n(x) a_n + (1-\lambda_n(x)) b_n) \in F(x)$, then $f(x) = c(x)$ is a Lipschitz selection for $F$ with the Lipschitz constant $M\left(\sum_{i=1}^{n} L_i\right) + nL$.

**Proof.** Let $x, y \in D$ and use the indication from Theorem 1. Since $f(x) - f(y) = (\lambda_1(x) a_1 + (1-\lambda_1(x)) b_1 - \lambda_1(y) a'_1 - (1-\lambda_1(y)) b'_1, \ldots, \lambda_n(x) a_n + (1-\lambda_n(x)) b_n - \lambda_n(y) a'_n - (1-\lambda_n(y)) b'_n)$, we have the inequality $\varrho(f(x), f(y)) \leq \sum_{i=1}^{n} |\lambda_i(x) a_i + (1-\lambda_i(x)) b_i - \lambda_i(y) a'_i - (1-\lambda_i(y)) b'_i| = P$. If we add to each of summands of

$P - \lambda_i(y)a_i + \lambda_i(y)a_i - (1-\lambda_i(y))b_i + (1-\lambda_i(y))b_i$ $(i=1, \ldots, n)$ we have $P \leqslant \sum_{i=1}^{n}(|\lambda_i(x) - \lambda_i(y)|\,|b_i - a_i| + \lambda_i(y)|a_i - a'_i| + (1-\lambda_i(y))\,|b_i - b'_i|) \leqslant$

$$\sum_{i=1}^{n}(L_i\delta(x, y)M + \lambda_i(y)L\delta(x, y) + (1-\lambda_i(y))L\delta(x, y))$$

$$= \left(M\left(\sum_{i=1}^{n} L_i\right) + nL\right)\delta(x, y). \text{ Q.E.D.}$$

**Lemma 3.** Let $F$: $D \to C(R^n)$ be a Lipschitz function. If $B$ is a bounded set in $D$, then $\bigcup_{x \in B}\{F(x)\}$ is bounded in $R^n$.

**Proof.** Let $a \in D$. Obviously it is sufficient to prove the statement of Lemma 3 for bounded sets of the form $B_r = \{x \in D: \delta(x, a) \leqslant r\}$, $r > 0$. According to the hypothesis $d(F(x), F(y)) \leqslant L\delta(x, y)$. Since $F(a) \in C(R^n)$ the set $F(a)$ is bounded in $R^n$, hence there exists $k > 0$ such that $u \in F(a)$ implies $\varrho(u, 0) \leqslant k$. Suppose $y \in F(x)$, $x \in B_r$, hence $\delta(x, a) \leqslant r$. We have $\varrho(y, F(a)) \leqslant d(F(x), F(a)) \leqslant L\delta(x, a) \leqslant Lr$. There exists $v \in F(a)$ such that $\varrho(y, F(a)) = \varrho(y, v)$, consequently $\varrho(y, 0) \leqslant \varrho(y, v) + \varrho(v, 0) = \varrho(y, F(a)) + \varrho(v, 0) \leqslant Lr + k$ and $\bigcup_{x \in B_r}\{F(x)\} \subset \bar{K}(\{0\}, Lr + k)$. Q.E.D.

In the following we shall deal with multifunctions of one real variable. Moreover, the metric $\varrho$ is supposed to be indicated by a norm.

**Theorem 3.** Let $F$: $\langle a, b\rangle \to C(R^n)$ be a Lipschitz function with the Lipschitz constant $L$. Then for any $x \in F(a)$ and any $y \in F(b)$ there exist Lipschitz selections $f, g$: $\langle a, b\rangle \to R^n$ for $F$ such that $f(a) = x$, $g(b) = y$ and with the same Lipschitz constant $L$.

**Proof.** We shall show the existence of $f$. The proof for $g$ is analogical. Let $y_0 \in F(a)$. According to Lemma 1 there exists $y_1^0 \in F(b)$ such that $\varrho(y_0, y_1^0) \leqslant L(b-a)$. Let $f_0$: $\langle a, b\rangle \to R^n$ be the linear function such that $f_0(a) = y_0$ and $f_0(b) = y_1^0$. We will construct a family of piecewise linear functions $f_n$, $n = 1, 2, \ldots$. Let $y_0^n = y_0 \in F(a)$. According to Lemma 1 there exists a point $y_1^n \in F(a + (b-a)2^{-n})$ such that $\varrho(y_0^n, y_1^n) \leqslant L(b-a)2^{-n}$. Having constructed a point $y_i^n \in F(a + i(b-a)2^{-n})$, $i < 2^n$, we can show using Lemma 1, a point $y_{i+1}^n \in F(a + (i+1)(b-a)2^{-n})$ such that $\varrho(y_i^n, y_{i+1}^n) \leqslant L(b-a)2^{-n}$. Hence we can get $2^n + 1$ of points $y_0^n, y_1^n, \ldots, y_{2^n}^n$ such that $y_i^n \in F(a + i(b-a)2^{-n})$ and $\varrho(y_i^n, y_{i+1}^n) \leqslant L(b-a)2^{-n}$ $(0 \leqslant i < 2^n)$. Let $f_n$: $\langle a, b\rangle \to R^n$ be such a piecewise linear function that $f_n(a + i(b-a)2^{-n}) = y_i^n$ and for $x \in (a + i(b-a)2^{-n}, a + (i+1)(b-a)2^{-n})$, $x = a + i(b-a)2^{-n} + t(b-a)2^{-n}$, $t \in (0, 1)$, let $f_n(x) = y_i^n + t(y_{i+1}^n - y_i^n)$.

We show that for each $n = 1, 2, \ldots$ the function $f_n$ has the following property $P$: $\varrho(f_n(x), f_n(y)) \leqslant L|x - y|$ for every $x, y \in \langle a, b\rangle$. Put $x_i = a + i(b-a)2^{-n}$, $i = 0, 1, \ldots, 2^n$. Hence $f_n(x_i) = y_i^n$, and $f_n(x_i + t(x_{i+1} - x_i)) = y_i^n + t(y_{i+1}^n - y_i^n)$, $t \in (0, 1)$. If $x, y \in \langle x_i, x_{i+1}\rangle$, $x = x_i + t_1(x_{i+1} - x_i)$, $y = x_i + t_2(x_{i+1} - x_i)$, $t_1 < t_2$, then a simple computation gives us $\varrho(f_n(x), f_n(y)) = (t_2-t_1)\varrho(y^n_{i+1}, y^n_i) \leqslant L(t_2-t_1)|x_{i+1}-x_i| = L|y-x|$. If $x_0<x_1<\ldots<x_i\leqslant x\leqslant x_{i+1}<\ldots<x_j\leqslant y\leqslant x_{j+1}<\ldots<x_{2^n}$, then $\varrho(f_n(x), f_n(x_{i+1})) \leqslant L|x_{i+1}-x|$, $\varrho(f_n(x_{i+1}), f_n(x_{i+2})) \leqslant L|x_{i+2}-x_{i+1}|$, ..., $\varrho(f_n(x_j), f_n(y)) \leqslant L|y-x_j|$. From the above inequalities it follows that $\varrho(f_n(x), f_n(y)) \leqslant \varrho(f_n(x), f_n(x_{i+1})) + \varrho(f_n(x_{i+1}), f_n(x_{i+2})) + \ldots + \varrho(f_n(x_j), f_n(y)) \leqslant L(|x_{i+1}-x|+|x_{i+2}-x_{i+1}|+\ldots+|y-x_j|) = L|x-x|$, and the property $P$ is proved.

According to Lemma 3 and the property $P$ the family of functions $f_n$, $n=1, 2, \ldots$, is uniformly bounded and equicontinuous on $\langle a, b\rangle$, hence it follows, according to Arzela—Ascoli theorem, that there exists a uniformly convergent subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$.

Put $f=\lim_{k\to\infty} f_{n_k}$. Obviously $\varrho(f(x), f(y)) \leqslant L|y-x|$. We show that $f$ is a required selection for $F$. Since $f_{n_k}(a+i(b-a)2^{-q})\in F(a+i(b-a)2^{-q})$ for $n_k>q$, we have $f(a+i(b-a)2^{-q})=\lim\limits_{k\to\infty} f_{n_k}(a+i(b-a)2^{-q}) \in F(a+i(b-a)2^{-q})$. Hence, $f(x)\in F(x)$ for each $x\in A=\{a+p(b-a)2^{-q}\colon p,\ q=0,\ 1,\ 2, \ldots$, and $0\leqslant p\leqslant 2^q\}$. We show that $f(y)\in F(y)$ for each $y\in\langle a, b\rangle - A$. Since $A$ is dense in $\langle a, b\rangle$, we can choose $x_i\in A$, $i=1, 2, \ldots$, such that $x_i\to y$. Suppose $f(y)\notin F(y)$. Hence there exists an $\varepsilon>0$ such that the $\varepsilon$-neighbourhood $K(f(y), \varepsilon)$ of $f(y)$ and $F(y)$ have the void intersection. It is easy to verify that if $x\in A$, $|y-x|<\frac{\varepsilon}{2L}$, then $F(x)\cap K\left(f(y), \frac{\varepsilon}{2}\right)=\emptyset$. On the contrary suppose $c\in F(x)\cap K\left(f(y), \frac{\varepsilon}{2}\right)$. Then $\varrho(c, F(y))>\frac{\varepsilon}{2}$ and $\max\{\max\limits_{a\in F(x)}\varrho(a, F(y)), \max\limits_{a\in F(y)}\varrho(a, F(x))\}>\frac{\varepsilon}{2}$, $d(F(x), F(y))>\frac{\varepsilon}{2}$. On the other hand, $d(F(x), F(y))\leqslant L\frac{\varepsilon}{2L}=\frac{\varepsilon}{2}$ — a contradiction. The relation $F(x)\cap K\left(f(y), \frac{\varepsilon}{2}\right)=\emptyset$, $x\in A$, implies the discontinuity of $f$ at $y$ because of $f(x_i)\in F(x_i)$. This is a contradiction with respect to the continuity of $f$. Hence $f$ is a required selection for $F$ with the Lipschitz constant $L$. Q.E.D.

**Corollary 2.** Let $F\colon R\to C(R^n)$ be a Lipschitz function with the Lipschitz constant $L$. Then there exists a Lipschitz selection for $F$ with the Lipschitz constant $L$.

**Proof.** The statement of Corollary 2 is an easy consequence of Theorem 3. It is sufficient to apply Theorem 3 in a suitable way on restrictions $F|\langle i, i+1\rangle$ and to use the fact $R=\bigcup\limits_{i=-\infty}^{\infty}\langle i, i+1\rangle$. Q.E.D.

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## SÚHRN

### O LITPSCHITZOVSKÝCH SELEKCIÁCH LIPSCHITZOVSKÝCH MULTIFUNKCIÍ

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V práci sa udávajú niektoré postačujúce podmienky k existencii lipschitzovskej selekcie lipschitzovskej multifunkcie $F: D \to C(R^n)$ ($D$ — metrický priestor; $R$ — reálna priamka; $C(R^n)$ — systém všetkých neprázdnych konvexných kompaktných podmnožín $R^n$, $n = 1, 2, \ldots$, opatrený Hausdorffovou metrikou). Dokazuje sa, napríklad, že keď $D$ je ľubovoľný metrický priestor a $n = 1, 2$, tak pre ľubovoľnú lipschitzovskú multifunkciu $F$ existuje lipschitzovská selekcia. Ďalej sa dokazuje, že pre každé $n = 1, 2, \ldots$ lipschitzovská multifunkcia $F: R \to (R^n)$ má lipschitzovskú selekciu. Uvádza sa taktiež závislosť liptschitzovskej konštanty selekcie od lipschitzovskej konštanty multifunkcie.

## РЕЗЮМЕ

### О СЕЛЕКЦИЯХ ЛИПШИЦА ДЛЯ МУЛТИФУНКЦИЙ ЛИПШИЦА

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В работе даются некоторые достаточные условия к существованию селекции Липшица для мултифункции Липшица $F: D \to C(R^n)$ ($D$ — метрическое пространство; $C(R^n)$ — система всех непустых бикомпактных подмножеств $R^n$, $n = 1, 2, \ldots$, снабжена метрикой Хаусдорффа). Например доказано, что если $D$ любое метрическое пространство и $n = 1$ или 2, то для любой мултифункции Липшица $F$ существует селекция Липшица. Далее показано, что для всякого $n = 1, 2, \ldots$ мултифункция Липшица $F: R \to C(R^n)$ имеет селекцию Липшица. Тоже дана зависимость константы Липшица селекции от константы Липшица мултифункции.