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MULTIPLES OF AN INTEGER IN THE PASCAL TRIANGLE

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The result of [2] is strenghtened according to which for every positive integer d the asymptotic density of those (x, y) for which d divides $\binom{x+y}{x}$ is equal to 1. Further, for every integer $m>0$, $n>0$, $a\geqq 0$, $b\geqq 0$ the asymptotical density of those u for which d divides $\binom{a+mu+b+nu}{a+mu}$ is equal to 1. This statement cannot be extended to $m=0$, $n=1$. The proofs are elementary and effective, and they mainly use the carries by the addition of p-adic representations of integers.

1. Introduction and notation

Let N denote the set of nonnegative integers. Small Latin letters will denote elements of N unless something else is explicitly said. Sequences will usually begin with the 0-th members. Beside the generally used mathematical notation we shall use the following notation for some subsets of $N\times N$.

Definition 1. a) M_n will denote the set of all (a, b) such that $a+b<n$.

b) $Sq(k, m, n)$ will denote the set of all (a, b) such that $m\cdot k\leqq a<(m+1)\cdot k$ and $n\cdot k\leqq b(n+1)\cdot k$.

c) Mult (d) will denote the set of all (a, b) such that $d\Big|\binom{a+b}{a}$.

Hence the first n lines of the Pascal triangle correspond to the function $F(x, y)=\binom{x+y}{x}$ defined on the set M_n. The sets $Sq(k, a, b)$ for positive k are special squares in $N\times N$. For $k>0$ fixed these squares form a partition of $N\times N$.

The set $N\times N$ can be imagined as a subset of the (euclidean) plane; the elements of $N\times N$ will correspond to the lattice points in the first quadrant. (The coordinate system must be turned if it ought to correspond to the Pascal triangle in the usual position.)

The cardinality of a set X will be denoted by card(X). The notion dens(X) of asymptotic density (or "natural density" by [1]) which is defined for $X \subseteq N$ will be extended to the subsets of $N \times N$ as follows.

Definition 2. For every subset X of $N \times N$

$$\mathrm{dens}_2(X) = \lim_{n \to \infty} \frac{\mathrm{card}(M_n \cap X)}{\mathrm{card}(M_n)} .$$

The index 2 can be usually omitted without any misunderstanding. If $X \subseteq N \times N$ and dens(X) exists then obviously $0 \leqq \mathrm{dens}(X) \leqq 1$ and $\mathrm{dens}(N \times N - X) = 1 - \mathrm{dens}(X)$. Further, if $X, Y \subseteq N \times N$ and $\mathrm{dens}(X) = \mathrm{dens}(Y) = 1$ then $\mathrm{dens}(X \cap Y) = 1$.

Remark. Let the members of $N \times N$ be ordered into the sequence

$$(0, 0),\ (0, 1),\ (1, 0),\ (0, 2),\ (1, 1),\ (2, 0),\ (0, 3),\ (1, 2),\ \ldots$$

(where (a, b) is before (c, d) if $a + b < c + d$ or $a + b = c + d$ and $a < c$). Let C be the mapping of $N \times N$ onto N such that every pair (x, y) is the $C(x, y)$-th member of the above sequence. Then for every $X \subseteq N \times N$

$$\mathrm{dens}_2(X) = \mathrm{dens}(\{C(x, y); (x, y) \in X\}) ,$$

where dens denotes the usual asymptotical density. If the function C is naturally used for the transformation of the Pascal triangle, then the obtained sequence will be

$$\binom{0}{0}, \binom{1}{0}, \binom{1}{1}, \binom{2}{0}, \binom{2}{1}, \binom{2}{2}, \binom{3}{0}, \binom{3}{1}, \ldots$$

If $p > 1$ and $0 \leqq ap^t$ then $(a_{t-1}a_{t-2} \ldots a_1 a_0)_p$ will denote the p-adic representation of a; zeros are added to the left if necessary. Further, let $\mathrm{Car}(a, b, p, i)$ be equal to 1 if a carry from the $(i-1)$-th position to the i-th position occurs by the addition of the p-adic representation of a, b; otherwise let $\mathrm{Car}(a, b, p, i) = 0$. The importance of the function Car for the next consideration will be clear from the following lemma, where also a formula is contained for Car.

Lemma 1. Let p be a prime. Then the maximal s such that $p^s \Big| \binom{a+b}{a}$ is given by

$$s = \sum_{i=1}^{\infty} \mathrm{Car}(a, b, p, i) ,$$

where

$$\mathrm{Car}(a, b, p, i) = \left[\frac{a+b}{p^i}\right] - \left[\frac{a}{p^i}\right] - \left[\frac{b}{p^i}\right] .$$

Proof. Since $\binom{a+b}{a} = \frac{(a+b)!}{a!\,b!}$, Theorem 4.2 from [1] gives

$$s = \sum_{i=1}^{\infty} \left[\frac{a+b}{p^i}\right] - \sum_{i=1}^{\infty} \left[\frac{a}{p^i}\right] - \sum_{i=1}^{\infty} \left[\frac{b}{p^i}\right] =$$

$$= \sum_{i=1}^{\infty} \left(\left[\frac{a+b}{p^i}\right] - \left[\frac{a}{p^i}\right] - \left[\frac{b}{p^i}\right] \right) = \sum_{i=1}^{\infty} \mathrm{Car}(a, b, p, i) .$$

Notice that the infinite sums can be replaced by finite ones; if $a, b < p^t$ then ∞ can be replaced by t.

2. The density of Mult(d)

D. Singmaster stated in [2, Theorem 1] that for any positive integer d, almost all binomial coefficients are divisible by d. He proved this statement for four different meanings of the words "almost all" (the first one corresponds to the dens_2 defined above). Here we prove the same result for another meaning of "almost all"; our result includes e.g. the first two cases from [2].

Theorem 1. For every positive integer d and every positive real ε there exists a real r such that for every bounded convex subset X of the first quadrant which contains a circle with the radius r it holds

$$\frac{\mathrm{card}(\mathrm{X} \cap \mathrm{Mult}(d))}{\mathrm{card}(\mathrm{X} \cap \mathrm{N} \times \mathrm{N})} > 1 - \varepsilon .$$

Hence the set Mult(d) has the asymptotical density 1 in a very uniform way. For example, the set

$$\mathrm{Y} = \{(a, b) \in \mathrm{N} \times \mathrm{N};\ a + b \geqq ([\sqrt{a+b}])^2 + \log_2 (a+b)\}$$

has the asymptotical density 1 in the sense $\mathrm{dens}_2(\mathrm{Y}) = 1$, and also in the sense

$$\lim_{n\to\infty} \frac{\mathrm{card}(\mathrm{Sq}(n, 0, 0) \cap \mathrm{Y})}{\mathrm{card}\,(\mathrm{Sq}(n, 0, 0))} = 1$$

(which corresponds to the second meaning of "almost all" in [2]). However, we cannot replace Mult(d) by Y in Theorem 1 above because there are arbitrarily large circles in the first quadrant which are disjoint with Y.

If $d = d_1 d_2 \ldots d_k$, where d_i are powers of distinct primes, then

$$\mathrm{Mult}(d) = \bigcap_{i=1}^{k} \mathrm{Mult}(d_i) .$$

Therefore (analogously as in [2]) it will be sufficient to prove Theorem 1 only for the case when d is a prime power. We prepare some lemmas which will be useful for this case.

Lemma 2. If p is a prime, $a<p^t$, $b<p^t$ and $p^s \Big| \binom{a+b}{a}$ then

$$p^s \Big| \binom{a+m\cdot p^t+b+n\cdot p^t}{a+m\cdot p^t}. \tag{2.1}$$

Proof. Since $p^s \Big| \binom{a+b}{a}$, Lemma 1 implies

$$\sum_{i=1}^{\infty} \mathrm{Car}(a, b, p, i) \geqq s .$$

However, since a, $b<p^t$ the symbol ∞ can be replaced by t. For every $i\leqq t$ it obviously holds that

$$\mathrm{Car}(a+m\cdot p^t, b+n\cdot p^t, p, i)=\mathrm{Car}(a, b, p, i) .$$

Therefore

$$\sum_{i=1}^{\infty} \mathrm{Car}(a+m\cdot p^t, b+n\cdot p^t, p, i)\geqq \sum_{i=1}^{t} \mathrm{Car}(a, b, p, i) \geqq s ,$$

which implies (2.1).

As an immediate consequence we obtain:

Lemma 3. If p is a prime and $t\geqq s$ then

$$\mathrm{card}(\mathrm{Sq}(p^t, m, n)-\mathrm{Mult}(p^s))\leqq \mathrm{card}(\mathrm{Sq}(p^t, 0, 0)-\mathrm{Mult}(p^s)) .$$

Lemma 4. For every prime p, every positive integer s and every positive real ε there is an integer $t\geqq s$ such that

$$\mathrm{card}(\mathrm{Sq}(p^t, 0, 0)-\mathrm{Mult}(p^s))<\frac{\varepsilon}{2}\cdot p^{2t} . \tag{4.1}$$

Lemma 4 is a consequence of Theorem 1(B) from [2], and therefore its proof will be omitted.

Proof of Theorem 1. We may restrict ourselves to the case $d=p^s$, where p is a prime. Let $d=p^s$ and $\varepsilon>0$ be given. Find an integer $t\geqq s$ which satisfies the inequality (4.1). Then we can put $r=9\cdot p^t$ in Theorem 1. To show that, consider arbitrary bounded convex subset X of the first quadrant which contains a circle with the radius r. Further in this proof let "a special square" mean a set of the form

$$\{(x, y)\in \mathbf{R}\times\mathbf{R};\ m\cdot p^t\leqq x<(m+1)\cdot p^t \wedge n\cdot p^t\leqq y<(n+1)\cdot p^t\} ,$$

where $m, n\in \mathbf{N}$ and $\mathbf{R}$ denotes the set of reals. The special squares form a partition of the first quadrant, and their intersections with $N \times N$ are the sets $\mathrm{Sq}(p^t, m, n)$. Now consider the smallest rectangle consisting of special squares which contains the set X. Let the lengths of its sides be $A \cdot p^t$, $B \cdot p^t$; we may assume that $A \geqq B$. Let X contain P special squares and have nonempty intersection with further Q ones.

Since X is covered by $P+Q$ special squares each of which contains less than $\frac{\varepsilon}{2} \cdot p^{2t}$ elements of $M = N \times N - \mathrm{Mult}(d)$, we have

$$\mathrm{card}(X \cap M) \leqq (P+Q) \cdot \frac{\varepsilon}{2} \cdot p^{2t} .$$

Since X contains P special squares each of which contains p^{2t} elements of $N \times N$, we have $\mathrm{card}(X \cap N \times N) \geqq P \cdot p^{2t}$. Therefore

$$\frac{\mathrm{card}(X \cap \mathrm{Mult}(d))}{\mathrm{card}(X \cap N \times N)} = 1 - \frac{\mathrm{card}(X \cap M)}{\mathrm{card}(X \cap N \times N)} > 1 - \frac{P+Q}{2P} \cdot \varepsilon .$$

It remais to prove $P+Q \leqq 2P$, i.e. $P \geqq Q$. We can easily see $Q \leqq 2A + 2B \leqq 4A$. On the other side, X contains two triangles with the same base at least $2r$, and with the sum of hights at least $(A-2) \cdot p^t$, hence with the common area at least $r \cdot (A-2) \cdot p^t$. Since the area of X is at most $(P+Q) \cdot p^{2t}$, and $A \cdot p^t \geqq 2r \geqq 18 \cdot p^t$, we have

$$(P+Q) \cdot p^{2t} \geqq 9 \cdot (A-2) \cdot p^{2t} = 8A \cdot p^{2t} + (A-18) \cdot p^{2t} \geqq 2Q \cdot p^{2t} ,$$

and hence $P \geqq Q$, which completes the proof.

In the next theorem we shall need the following notation:

Definition 3. For every subset X of the (euclidean) plane $R \times R$ and every positive real r denote

$$X(r) = \left\{ (x, y) \in R \times R ; \left(\frac{x}{r}, \frac{y}{r} \right) \in X \right\} .$$

The next theorem can be easily obtained from Theorem 1.

Theorem 2. Let X be the union of finitely many bounded convex subsets of the first quadrant with nonempty interior. Then for every positive integer d

$$\lim_{r \to \infty} \frac{\mathrm{card}(X(r) \cap \mathrm{Mult}(d))}{\mathrm{card}(X(r) \cap N \times N)} = 1 .$$

Theorems 1(A) and 1(B) from [2] can be obtained from the above theorem if X is the triangle with the vertices (0, 0), (0, 1), (1, 0) or the square with the vertices (0, 0), (0, 1), (1, 0), (1, 1), respectively.

3. Intersections of Mult(d) with half lines

Consider a half-line P with the end-point and a further point belonging to $N \times N$. Let there be (uniquely determined) nonnegative integers a, b, m, n, $m+n \neq 0$, $D(m, n)=1$ such that $u \mapsto (a+mu, b+nu)$ is a bijection between N and $P \cap (N \times N)$. Let U be the set which corresponds to the set $P \cap \mathrm{Mult}(d)$ in this bijection. We shall prove that dens(U) = 1 whenever both m, n are positive (i.e., P is not parallel with any of the coordinate axes).

Theorem 3. Let a, b, s be nonnegative integers, m, n be positive integers, p be a prime and

$$U = \left\{ u ; p^s \,\middle|\, \binom{a+mu+b+nu}{a+mu} \right\}.$$

Then dens(U) = 1.

Proof. Since dens(U) can only decrease if s increases we may assume $s \geqq (m+n)^2 + (a+b)$ without loss of generality. Denote $r = s(s+1)$. For any w, $0 < w < p^r$, denote by S(w) the set of all $u \in N$ such that the p-ary representative $(w_{r-1} w_{r-2} \ldots w_1 w_0)_p$ of w occurs as a segment in the p-ary representation of u (with additional zeros on the left if necessary).

It holds dens(S(w)) = 1 for every w. Indeed, consider a positive integer t. For an arbitrary u divide the last rt ciphers of u into t segments of the length r. If $u \in N - S(w)$ then no of these segments can be $(w_{r-1} w_{r-2} \ldots w_1 w_0)_p$, and hence there are at most $(p^r - 1)^t$ possibilities for the last rt ciphers of any $u \in N - S(w)$. This easily implies

$$\mathrm{card}\{x \in N - S(w) ; x < n\} < 2n \cdot (1 - p^{-r})^t$$

for every $n > p^{rt}$. Now $\lim_{t \to \infty} (1 - p^{-r})^t = 0$ easily implies dens(N − S(w)) = 0 which also gives dens(S(w)) = 1.

If we find w such that $S(w) \subseteq U$ then dens(S(w)) = 1 will imply dens(U) = 1, and the proof will be finished.

To find such a w, denote $v = \left[\frac{p^s - 1}{m+n}\right] + 1$ and consider arbitrary u of the form

$$u = x \cdot p^{j+s} + v \cdot p^j + y \tag{3.1}$$

where $0 \leqq y < p^j$ and $j \geqq s$. (Hence the p-ary representation of u contains the segment $(v_{s-1} v_{s-2} \ldots v_0)_p$ which ends at its j-th position.) We shall show $\mathrm{Car}(a + mu, b + nu, p, j+s) = 1$. Since

$$a + mu = mx \cdot p^{j+s} + mv \cdot p^j + y + a \geqq mx \cdot p^{j+s}$$

and

$$a+mu-mx\cdot p^{j+s}<mv\cdot p^j+p^j+p^s\leqq p^j(mv+2)=$$
$$=p^j((m+n)v-nv+2)<p^j(p^s+m+n-nv+2)<p^{j+s}$$

we have $\left[\frac{a+mu}{p^{j+s}}\right]=mx$. Analogously $\left[\frac{b+nu}{p^{j+s}}\right]=nx$. However, $(m+n)v\geqq p^s$, and hence

$$\left[\frac{a+mu+b+nu}{p^{j+s}}\right]\geqq\left[mx+nx+\frac{(m+n)v\cdot p^j}{p^{j+s}}\right]\geqq mx+nx+1\,.$$

Therefore $\mathrm{Car}(a+mu,\ b+nu,\ p,\ j+s)=1$.

Now put $w=v\cdot(p^s+p^{2s}+\ldots+p^{s^2})$; the p-ary representation $(w_{r-1}w_{r-2}\ldots w_1w_0)_p$ of w consists of s-times repeated $(v_{s-1}v_{s-2}\ldots v_1v_0)_p$ and s additional zeros at the end. If $u\in\mathrm{S}(w)$ then u can be represented in the form (3.1) for s different values of j; and hence

$$\sum_{i=1}^{\infty}\mathrm{Car}(a+mu,\ b+nu,\ p,\ i)\geqq s$$

which implies $u\in\mathrm{U}$. Hence $\mathrm{S}(w)\subseteq\mathrm{U}$ which completes the proof of Theorem 3.

The next theorem can be proved in the same way as Theorem 2.

Theorem 4. Let a, b be nonnegative integers, m, n, d be positive integers. Then the asymptotic density of the set

$$\mathrm{U}=\left\{u\,;\,d\,\middle|\,\binom{a+mu+b+nu}{a+mu}\right\}$$

is equal to 1.

The assumption that m, n are positive cannot be omitted in Theorem 3 and Theorem 4, as the following theorem shows.

Theorem 5. Let p be a prime, a be a nonnegative integer and $(a_{k-1}a_{k-2}\ldots a_1a_0)_p$ be the p-adic representation of a. Then

$$\mathrm{dens}\left(\left\{u;\,p\,\middle|\,\binom{a+u}{a}\right\}\right)=1-\prod_{i=0}^{k-1}\left(1-\frac{a_i}{p}\right).$$

Proof. The main idea will be indicated. Denote $\mathrm{U}=\left\{u\,;\,p\,\middle|\,\binom{a+u}{a}\right\}$. Then $u\in\mathrm{N}-\mathrm{U}$ if and only if $\mathrm{Car}(a,\ u,\ p,\ i)=0$ for all i. The latest conditions is equivalent to $u_i+a_i<p$ for all $i<k$, where u_i is the i-th cipher in the p-ary representation of u. This inequality has $p-a_i$ solutions for the unknown u_i. Therefore there are $\prod_{i-0}^{k-1}(p-a_i)$ possibilities for the last k ciphers of (the p-adic representation of) the elements of $\mathrm{N}-\mathrm{U}$. Hence there are $p^k-\prod_{i=0}^{k-1}(p-a_i)$

possibilities for the last k ciphers of the elements of U. From that the formula for dens(U) can be obtained.

At the end notice that for every a, b, n, d the asymptotic density of the set $U = \left\{u\,;\, d \,\middle|\, \binom{a+b+nu}{a}\right\}$ exists, and it can be computed because $u \equiv v(\text{mod } d \cdot a!)$ implies $u \in U \leftrightarrow v \in U$. Hence

$$\text{dens}(U) = \frac{1}{d \cdot a!} \cdot \text{card}\left(\left\{u < d \cdot a!\,;\, d \,\middle|\, \binom{a+b+nu}{a}\right\}\right).$$

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SÚHRN

NÁSOBKY ĽUBOVOĽNÉHO ČÍSLA V PASCALOVOM TROJUHOLNÍKU

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Označme Mult(d) množinu všetkých usporiadaných dvojíc $(a, b) \in N \times N$, pre ktoré je binomický koeficient $\binom{a+b}{a}$ násobkom čísla d. Výsledok D. Singmastera, že množina Mult(d) má asymptotickú hustotu rovnajúcu sa 1, sa zosilňuje nasledovne: Pre každé prirodzené číslo d a každé kladné reálne číslo ε existuje také kladné číslo r, že pre každú konvexnú množinu, ktorá obsahuje kruh polomeru r platí nerovnosť

$$\text{card}(X \cap \text{Mult}(d)) > (1-\varepsilon) \cdot \text{card}(X \cap (N \times N))$$

Ďalej sa dokazuje, že pre každé a, $b \in N$, m, n, $s \in N - \{0\}$ je asymptotická hustota množiny

$$U = \{u \in N\,;\, (a+mu, b+nu) \in \text{Mult}(d)\}$$

rovná 1. Tento výsledok sa nedá zovšeobecniť na prípad $m = 0$; v práci sa počíta hustota množiny U pre $m = 0$, d prvočíslo.

РЕЗЮМЕ

КРАТНЫЕ ЛЮБОГО ЧИСЛА В ТРЕУГОЛЬНИКЕ ПАСКАЛЯ

Иван Корец, Братислава

Пусть $N = \{0, 1, 2, \ldots\}$ и $\mathrm{Mult}(d)$ обозначает множество всех пар $(a, b) \in N \times N$ для которых биномиальный коэффициент

$$\binom{a+b}{a}$$

делится на d. Результат Д. Сингмастера, что для любого натурального числа d асимптотическая плотность множества $\mathrm{Mult}(d)$ равна единице, усиливается следующим образом. Для любого натурального числа d и положительного вещественного числа ε существует такое положительное число r, что для всякой выпуклой области X, которая содержит круг радиуса r выполнияется неравенство

$$\mathrm{card}(X \cap \mathrm{Mult}(d)) > (1-\varepsilon) \cdot \mathrm{card}(X \cap (N \times N))$$

Далее доказывается, что для всех $a, b \in N$, $m, n, d \in N \in N - \{0\}$ асимптотическая плотность множества $\{u \in N;\ (a + mu, b + nu) \in \mathrm{Mult}(d)\}$ равна единице.