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COMPLEMENTED P-ALGEBRAS

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An algebra $L = (L; \vee, \wedge, {}^*, 0, 1)$ is called a *p-algebra* (or a *pseudocomplemented* lattice) if $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and $x \leqslant a^*$ iff $a \wedge x = 0$. A p-algebra is said to be *nontrivial* if $x^* \neq 0$ whenever $x \neq 1$.

It is known in Lambrou [4] using the axiom of choice and working with ultrafilters that a nontrivial p-algebra is complemented. We give a short elementary proof of this fact (Theorem 1). In special cases complemented p-algebras are Boolean algebras. In Theorem 2 we characterize those equational classes of p-algebras in which that is the case.

Theorem 1. A p-algebra L is complemented if and only if L is nontrivial.

Proof. Since $x' \leqslant x^*$, we see that a complemented p-algebra is nontrivial. Conversely, let L be nontrivial. Evidently, $x \leqslant y$ yields $x^* \geqslant y^*$. Therefore, $(x \vee x^*)^* \leqslant x^* \wedge x^{**} = 0$, and this implies $x \vee x^* = 1$. Hence x^* is a complement of x in L.

Pentagon, the five-element nonmodular lattice, is the smallest non-Boolean complemented p-algebra. It is known that the p-algebras can be defined in terms of identities, that means the class of all p-algebras is equational (cf. [1]).

Theorem 2. Let $\boldsymbol{K}$ be an equational subclass of the class of all p-algebras. The following conditions are equivalent:

(i) $\boldsymbol{K}$ does not contain a pentagon;

(ii) Every algebra from $\boldsymbol{K}$ satisfies identity

$$x = x^{**} \wedge (x \vee x^*);$$

(iii) In $\boldsymbol{K}$, every complemented algebra is a Boolean one.

Proof. (i) $\Rightarrow$ (ii). In [2, Theorems 4 and 6] we have shown that the equational subclasses satisfying (i) are contained in the class of p-algebras defined by identity $x = x^{**} \wedge (x \vee x^*)$. (ii) $\Rightarrow$ (iii). Since $x \vee x^* = 1$ in a complemented p-algebra, we see that $x = x^{**}$ for every x. But the "closed" elements form a Boolean algebra (see [1, Theorem 6.4]). (iii) $\Rightarrow$ (i) is trivial.

Remark. In view of Theorem 2 we can give a different proof of Theorem in [3]: A complete lattice with 0 and 1 is an atomic Boolean algebra if and only if it is semisimple (i.e. the intersection of all maximal ideals of L is $\{0\}$) and completely distributive. Evidently, L is distributive, pseudocomplemented and dually pseudocomplemented, by complete distributivity. (Let a^+ denote the dual pseudocomplement of a, i.e. $a \vee x = 1$ iff $x \geq a^+$.) Then $x = x^{++} \vee (x \wedge x^+)$ and, by [1, Lemma 15.5], every maximal ideal of L contains ideal $\{x \in L: x^+ = 1\}$ $= \{x \wedge x^+ : x \in L\}$. By semisimplicity, $x \wedge x^+ = 0$ for every $x \in L$. Hence L is complemented. Therefore, L is a Boolean algebra by Theorem 2. The rest follows from the Tarski's Theorem for Boolean algebras.

REFERENCES

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SÚHRN

KOMPLEMENTÁRNE p-ALGEBRY

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V práci sa charakterizujú tie p-algebry, ktoré sú aj komplementárnymi zväzmi.

РЕЗЮМЕ

p-АЛГЕБРЫ С ДОПОЛНЕНИЯМИ

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Охарактеризованы те p-алгебры, которые являются решеткой с дополнениями.