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ON A TECHNICAL LEMMA IN LATTICE ORDERED GROUPS

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The aim of the article is to prove the following computational lemma:

(A) Let G be a σ-complete lattice ordered group. Let $(a_{n,i,j})_{n,i,j}$ be a bounded sequence of elements of G such that $a_{n,i,j}\searrow O(j\to\infty,\ n,\ i=1, 2, \ldots)$. Then to every $a\in G$, $a>O$ there exists such a bounded sequence $(a_{i,j})_{i,j}$ that $a_{i,j}\searrow O$ $(j\to\infty,\ i=1, 2, \ldots)$ and such that for every t: $N\to N$

$$a\wedge\Big(\sum_{n=1}^{\infty}\bigvee_{i=1}^{\infty}a_{n,i,t(i+n)}\Big)\leqq\bigvee_{i=1}^{\infty}a_{i,t(i)}.$$

This lemma was discovered by D. H. Fremlin ([2]) in a connection with his simple proof (see also [5]) of the famous Matthes—Wright vector lattice valued measure extension theorem ([2]). The lemma substitutes successfully the usual ε-technique in vector lattice valued analysis (see e.g. [2], [3], [4], [5], [9]) and it seems to be useful also in a more general l-group valued case ([7], [8], [9]). Of course, from an algebraic point of view it presents a quite special result and therefore it does not appear in monographs on ordered groups.

We present here two proofs of the mentioned assertion. The first one is presented in Part 2 and it uses a representation technique ([1]). By the same technique we prove that a regularity condition from the paper [6] holds in any lattice ordered group. Part 2 belongs to the second author. On the other hand, in Part 3 which belongs to the first author we present an elementary, purely algebraic proof. Part 1 contains some necessary notations and notions and, for the convenience of the reader, a proof of the assertion (A) in the vector lattice case. Namely vector lattice version is used in the first concept in Part 2.

1

By a σ-complete l-group (lattice ordered group) we mean a boundedly σ-complete lattice G being simultaneously a commutative group and satisfying the identity $a+(b\vee c)=(a+b)\vee(a+c)$. If G is, moreover, a (real) linear space satisfying the identity $\alpha(a\vee b)=(\alpha a)\vee(\alpha b)$ for $\alpha\in R$, $\alpha>0$, a, $b\in G$, then G is called a vector lattice or a Riesz space. A vector lattice is called to be boundedly σ-complete, if every bounded sequence in the space has the supremum.

1.1. Proposition ([2] lemma 1C, [5] proposition 3). Proposition (A) holds in any boundedly σ-complete vector lattice. Moreover, $a_{i,j}=a\wedge b_{i,j}$, where $b_{i,j}=\bigvee_{r=1}^{i-1} 2^r a_{r,i-r,j}$.

Proof. Evidently $b_{i+n,j}\geqq 2^n a_{n,i,j}$, whereas $a_{n,i,j}\leqq 2^{-n}b_{i+n,j}$, hence

$$\sum_{n=1}^{k}\bigvee_{i=1}^{\infty} a_{n,i,t(i+n)}\leqq\sum_{n=1}^{k}2^{-n}\left(\bigvee_{i=1}^{\infty} b_{i+n,t(i+n)}\right)\leqq$$

$$\leqq\left(\sum_{n=1}^{k}2^{-n}\right)\bigvee_{j=1}^{\infty} b_{j,t(j)}\leqq\bigvee_{j=1}^{\infty} b_{j,t(j)}.$$

Since every vector lattice is distributive, we have

$$a\wedge\left(\sum_{n=1}^{k}\bigvee_{i=1}^{\infty} a_{n,i,t(i+n)}\right)\leqq\bigvee_{i=1}^{\infty}(a\wedge b_{i,t(i)})=\bigvee_{i=1}^{\infty} a_{i,t(i)}.$$

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2.1. Proposition. Proposition (A) holds in any σ-complete l-group G.

Proof. By [1] theorem 4 there exist a vector lattice F and an l-group isomorphism h: $G\to h(G)\subset F$ preserving all supremums and infimums. Therefore by Proposition 1.1

$$h\left(a\wedge\sum_n\bigvee_i a_{n,i,t(i+n)}\right)=h(a)\wedge\sum_n\bigvee_i h(a_{n,i,t(i+n)})\leqq$$

$$\leqq\bigvee_i h(a_{i,t(i)})=h\left(\bigvee_i a_{i,t(i)}\right),$$

since $h(a_{i,j})=h(a)\wedge\bigvee_{r=1}^{i-1}2^r h(a_{r,i-r,j})$. Finally,

$$a\wedge\sum_n\bigvee_i a_{n,i,t(i+n)}\leqq\bigvee_i a_{i,t(i)}$$

since h is an isomorphism.

By a similar technique we prove that every lattice ordered group is weakly regular (see [6], cf. also [10]).

2.2. Proposition. Let G be a σ-complete l-group. Let $(a_{i,j})$, $a_{i,j}\searrow O(j\to\infty)$, $(i, j=1, 2, \ldots)$ $a\in G$, $a>O$. Then there exists t: $N\to N$ such that for every n, $\sum_{i=1}^{n} a_{i,t(i)}\ngeqq a$.

Proof. By [1] theorem 4 there exist a vector lattice F and an isomorphism $h\colon G \to h(G) \subset F$. Moreover, F consists of almost finite continuous functions on a compact, Hausdorff and extremally disconnected space E. Since $h(a_{i,j}) \searrow 0$, $(j \to \infty)$, we get that the set

$$A_i = \{x \in E;\ (h(a_{i,j})(x))_j \not\searrow 0\}$$

is of the first category. Since $a > O$, we have $h(a) > 0$, i.e. there exists a clopen set U and an $\varepsilon > 0$ such that $h(a)(x) > \varepsilon$ for every $x \in U$. Since U is open, it is not of the first category, hence there exists $x_0 \in U \setminus \bigcup_i A_i$. So $h(a_{i,j})(x_0) \searrow 0$ $(j \to \infty)$ for every i. Therefore there is $t\colon N \to N$ such that

$$\sum_{i=1}^{n} h(a_{i,t(i)})(x_0) < \frac{\varepsilon}{2}$$

for every n, hence

$$\sum_{i=1}^{n} h(a_{i,t(i)}) \ngeq h(a), \quad n = 1, 2, \ldots$$

i.e.

$$\sum_{i=1}^{n} a_{i,t(i)} \ngeq a, \quad n = 1, 2, \ldots$$

3

In this section G is a σ-complete l-group. We write $1 \cdot c = c$ and $n \cdot c = (n-1) \cdot c + c$ for any $c \in G$ and any $n \in N$.

3.1. Lemma. If $c_i \in G$ $(i = 1, \ldots, 2^k)$, then

$$\sum_{j=1}^{2^k} c_j \leqq \bigvee_{j=1}^{2^k} 2^k c_j .$$

Proof. We can do it by the induction, the first step of it being the following:

$$O \leqq (c_1 - c_2) \vee (c_2 - c_1)$$

$$c_1 + c_2 \leqq (c_1 + c_2) + (c_1 - c_2) \vee (c_2 - c_1) = (c_1 + c_1) \vee (c_2 + c_2).$$

3.2. Lemma. If $b \wedge (2^k b_i) \leqq a_i$ $(i = 1, \ldots, n)$, then $b \wedge \left(2^k \bigvee_{i=1}^{n} b_i\right) \leqq \bigvee_{i=1}^{n} a_i$.

Proof. By Lemma 3.1

$$b \wedge \left(2^k \bigvee_i b_i\right) = b \wedge \left(\bigvee_{i_1=1}^{n} \cdots \bigvee_{i_{2^k}=1}^{n} \sum_{j=1}^{2^k} b_{i_j}\right) \leqq$$

$$\leqq b \wedge \left(\bigvee_{i_1=1}^{n} \dots \bigvee_{i_{2^k}=1}^{n} \bigvee_{j=1}^{2^k} 2^k b_{i_j} \right) =$$

$$= \bigvee_{i_1=1}^{n} \dots \bigvee_{i_2k=1}^{n} \bigvee_{j=1}^{2^k} (b \wedge 2^k b_{i_j}) \leqq \bigvee_{i=1}^{n} a_i .$$

3.3. Lemma. If $b \wedge (2^k c_k) \leqq c$ $(k = 1, \dots, n)$, then $b \wedge \left(\sum_{k=1}^{n} c_k \right) \leqq c$.

Proof. By Lemma 3.1 (with $k = 1$) we get

$$b \wedge \left(\sum_{k=1}^{n} c_k \right) = b \wedge \left(c_1 + \sum_{k=2}^{n} c_k \right) \leqq b \wedge \left((c_1 + c_1) \vee \left(\sum_{k=2}^{n} c_k + \sum_{k=2}^{n} c_k \right) \right) \leqq$$

$$\leqq (b \wedge 2c_1) \vee \left(b \wedge \left(2c_2 + \sum_{k=3}^{n} 2c_k \right) \right) \leqq$$

$$\leqq (b \wedge 2c_1) \vee \left(b \wedge \left((2c_2 + 2c_2) \vee \left(\sum_{k=3}^{n} 2c_k + \sum_{k=3}^{n} 2c_k \right) \right) \right) =$$

$$= (b \wedge 2c_1) \vee (b \wedge 2^2 c_2) \vee \left(b \wedge \left(2^2 c_3 + \sum_{k=4}^{n} 2^2 c_k \right) \right) \leqq \dots \leqq \bigvee_{k=1}^{n} b \wedge (2^k c_k) \leqq c .$$

3.4. Proposition. Proposition (A) holds in any σ-complete l-group G.

Prof. Put $a_{i,j} = a \wedge \left(\bigvee_{k=1}^{i} 2^k a_{k,\, i-k+1,\, j} \right)$. Evidently, $a_{i+k-1,\, t(i+k-1)} \geqq a \wedge 2^k a_{k,\, i,\, t(i+k-1)}$, so by Lemma 3.2

$$c = \bigvee_{i} a_{i,\, t(i)} \geqq a \wedge \left(2^k \bigvee_{i=1}^{n} a_{k,\, i,\, t(i+k-1)} \right).$$

Finally, by Lemma 3.3

$$a \wedge \sum_{k=1}^{n} \bigvee_{i=1}^{\infty} a_{k,\, i,\, t(i+k-1)} \leqq \bigvee_{i=1}^{\infty} a_{i,\, t(i)} .$$

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SÚHRN

O JEDNEJ TECHNICKEJ LEME VO ZVÄZOVO USPORIADANÝCH GRUPÁCH

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Dokazuje sa táto lema: Nech G je σ-úplná zväzovo usporiadaná grupa. Nech $(a_{n,i,j})_{n,i,j}$ je taká ohraničená postupnosť prvkov grupy G, že $a_{n,i,j} \searrow O$ ($j \to \infty$, n, $i = 1, 2, \ldots$). Potom k ľubovoľnému $a \in G$, $a > O$ existuje taká ohraničená postupnosť $(a_{i,j})_{i,j}$, že $a_{i,j} \searrow O$ ($j \to \infty$, $i = 1, 2, \ldots$) a taká, že pre každé t: $N \to N$ platí

$$a \wedge \left(\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,t(i+n)} \right) \leqq \bigvee_{i=1}^{\infty} a_{i,t(i)}.$$

Táto lema nahradzuje v G-hodnotovej analýze obvyklú epsilónovú techniku.

РЕЗЮМЕ

ОБ ОДНОЙ ТЕХНИЧЕСКОЙ ЛЕММЕ В СТРУКТУРНО УПОРЯДОЧЕННЫХ ГРУППАХ

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Доказывается следующая лемма: Пусть G – σ-полная структурно упорядоченная группа. Пусть $(a_{n,i,j})_{n,i,j}$ ограниченная последовательность элементов из G такая, что $a_{n,i,j} \searrow O$ ($j \to \infty$, n, $i = 1, 2, \ldots$). Тогда к любому $a \in G$, $a > O$ существует такая ограниченная последовательность $(a_{i,j})_{i,j}$, что $a_{i,j} \searrow O$ ($j \to \infty$, $i = 1, 2, \ldots$) и такая, что для всякого t: $N \to N$ справедливо

$$a \wedge \left(\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,t(i+n)} \right) \leqq \bigvee_{i=1}^{\infty} a_{i,t(i)}.$$

Эта лемма употребляется в анализе функций с значениями в G вместо обычной ε-ой техники.