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AUTOGENOMORPHISMS ON FINITE BOOLEAN ALGEBRAS

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Introduction

E. K. Blum and D. R. Estes [1] introduced and studied the concept of a genomorphism between algebras of arbitrary similarity type. It is a generalization of the homomorphism concept, which presupposes algebras of the same similarity type.

Blum and Estes [1] characterized autogenomorphisms on free semigroups and groups. In this paper we shall describe autogenomorphisms on finite Boolean algebras.

Basic concepts and notation

$A=(A;\alpha_1, \dots, \alpha_n)$ denotes a universal algebra with underlying set A and fundamental operations α_i: $A_i \to A$; $1 \leqslant i \leqslant n$. Having $\alpha=\{\alpha_1, \dots, \alpha_n\}$, we write $(A;\alpha)$ instead of $(A;\alpha_1, \dots, \alpha_n)$.

Let H be a nonempty subset of A. Then $[H;\alpha]$ denotes the subalgebra of $(A;\alpha)$ generated by H. When $H=\{c_1, \dots, c_k\}$, we write $[c_1, \dots, c_k;\alpha]$ or $[c_1, \dots, c_k]$ instead of $[H;\alpha]$.

Definition 1. (See [1]). Let $A=(A;\alpha)$ and $B=(B;\beta)$ be two arbitrary algebras. A mapping φ: $A \to B$ is said to be generative if for each operator α_i of arity $r_i > 0$

$$\varphi(\alpha_i(a_1, \dots, a_{r_i})) \in [\varphi(a_1), \dots, \varphi(a_{r_i});\beta].$$

The mapping φ is said to be congruential if $\varphi(a'_j)=\varphi(a_j)$, $1 \leqslant j \leqq r_i$, implies

$$\varphi(\alpha_i(a_1, \dots, a_{r_i}))=\varphi(\alpha_i(a'_1, \dots, a'_{r_i})).$$

A mapping φ is called a genomorphism, if it is both generative and congruential.

The prefixes mono, epi, iso, endo and auto have the same connotation as in the category of sets. Thus, for example, φ: $A \to A$ is an autogenomorphism if φ is a genomorphism that is injective (one-to-one) and surjective (onto).

Remark 1. Every homomorphism φ is a genomorphism, since then $\varphi(\alpha_i(a_1, \ldots, a_{r_i})) = \beta_i(\varphi(a_1), \ldots, \varphi(a_{r_i}))$. An injective generative mapping is obviously a genomorphism.

Lemma A. ([1], Lemma 2). Let φ: $A \to B$ be generative. Let $S \subseteq A$. Then $\varphi([S; \alpha]) \subseteq [\varphi(S); \beta]$.

Lemma B. ([1]), Lemma 3). The composition of two autogenomorphisms is again an autogenomorphism.

We suppose that the reader is familiar with basic notions and results on Boolean algebras. The following statement is routine (see [2]).

Lemma C. Let a, b, c be atoms of a Boolean algebra $B = (B; \vee, \wedge, ', 0, 1)$. Then $[a, b] = \{a, b, a', b', a \vee b, (a \vee b)', 0, 1\}$ and $[a, b, c] = \{a, b, c, a', b', c', a \vee b, a \vee c, b \vee c, (a \vee b)', (a \vee c)', (b \vee c)', a \vee b \vee c, (a \vee b \vee c)', 0, 1\}$.

Lemma D. Let B be a Boolean algebra and a, $b \in B$. Assume $a \wedge b = 0$. Then $[a, b] = \{0, 1, a, b, a', b', a' \wedge b', a \vee b\}$.

Proof. Let $FB(x, y)$ denote the free Boolean algebra on two generators x, y. There exists an epimorphism f: $FB(x, y) \to [a, b]$ with $f(x) = a$ and $f(y) = b$. Clearly $x \wedge y \in \operatorname{Ker} f$. Now the statement follows from $|FB(x, y)| = 16$ and the fact that $[a, b] \cong FB(x, y)/\operatorname{Ker} f$.

Main results

Theorem 1. Let φ be an autogenomorphism of a Boolean algebra B. Then $\varphi(\{0, 1\}) = \{0, 1\}$, and for each $a \in B$, $\varphi(a') = (\varphi(a))'$.

Proof. Let $\varphi(a) = 0$, $\varphi(b) = 1$, a, $b \in B$. By Lemma A, $\varphi([a, b]) \subseteq [0, 1]$, hence $\{a, b\} = \{0, 1\}$. Thus, $\varphi(\{0, 1\}) = \{0, 1\}$.

Suppose that a is an element of B. Therefore, $\varphi(a') \in [\varphi(a)] = [\varphi(a), (\varphi(a))', 0, 1]$, which implies $\varphi(a') = (\varphi(a))'$, by the previous result.

Corollary. Let $B = (\{a, a', 0, 1\}; \vee, \wedge, ', 0, 1)$ be a Boolean algebra. Let φ be an autogenomorphism on B. Then $\varphi(\{0, 1\}) = \{0, 1\}$ and $\varphi(\{a, a'\}) = \{a, a'\}$.

Now, we want to show that every autogenomorphism on a finite Boolean algebra associates each atom with an atom or a coatom. Evidently, this is true for Boolean algebras with four elements.

Lemma 1. Let φ be an autogenomorphism on a Boolean algebra B with $|B| \geqq 8$. Let $\varphi(a)$, $\varphi(b)$ be different atoms of B. Then one of the following cases occurs:

(i) $a < b$ or $b < a$,
(ii) $a \wedge b = 0$,
(iii) $a \vee b = 1$.

Proof: Evidently, 0, 1, a, b are mutually different. By Lemma D, $|[\varphi(a), \varphi(b)]| = 8$. Therefore, $|[a, b]| \leqslant 8$, by Lemma A. Denote by $FB(x, y)$ the free Boolean algebra on two free generators x and y. There is an epimorphism f: $FB(x, y) \to [a, b]$ such that $f(x) = a$ and $f(y) = b$. Clearly, $[a, b] \cong FB(x, y)$-/Ker f. Since $|FB(x, y)| = 16$, we have $|\text{Ker } f| \geqq 2$. Evidently, $x \wedge y$, $x' \wedge y$, $x \wedge y'$, $x' \wedge y'$ are the only atoms of $FB(x, y)$. Hence, $x \wedge y \in \text{Ker } f$ or $x \wedge y' \in \text{Ker } f$ or $x \wedge y' \in \text{Ker } f$ or $x' \wedge y' \in \text{Ker } f$, which gives one of the cases (i)—(iii).

Theorem 2. Let φ be an autogenomorphism on a Boolean algebra B and $\varphi(x) = y$ for some $x \in B$. Let φ': $B \to B$ be defined as follows:

$$\varphi'(z) = \begin{cases} y' & \text{for } z = x, \\ -y & \text{for } z = x', \\ \varphi(z) & \text{for } x \neq z \neq x'. \end{cases}$$

Then φ' is an autogenomorphism on B.

Proof. Evidently, $\varphi' = \psi \circ \varphi$, where

$$\psi(t) = \begin{cases} y' & \text{for } t = y, \\ y & \text{for } t = y', \\ t & \text{for } y \neq t \neq y'. \end{cases}$$

It is easy to verify that ψ is an autogenomorphism on B. It follows that φ' is also an autogenomorphism (see Lemma B) and the proof is complete.

Definition 2. Two autogenomorphisms φ_1, φ_2 on a Boolean algebra B are said to be *similar*, if for each $a \in B$, $\varphi_1(a) = \varphi_2(a)$ or $\varphi_1(a) = \varphi_2(a')$.

Remark 2. It is easy to see, that the similarity is an equivalence relation on the set of all autogenomorphisms on a Boolean algebra B.

Theorem 3. Let φ be an autogenomorphism on a Boolean algebra B with $|B| \geqq 8$. Let $\varphi(a)$, $\varphi(b)$ be different atoms of B. Then there exists a similar autogenomorphism ψ such that

(i) $c \wedge d = 0$ for $\psi(c) = \varphi(a)$ and $\psi(d) = \varphi(b)$,

(ii) $\psi(z) = \varphi(z)$ for $c \neq z \neq d$ and $c' \neq z \neq d'$.

Proof. In accordance with Lemma 1 we can choose $c = a$ and $d = b'$ or $c = b$ and $d = a'$ in the first case, and $c = a'$, $d = b'$ in the third case. The rest of the proof is straightforward.

Lemma 2. Let φ be an autogenomorphism on a Boolean algebra B with $|B| \geqq 8$. Let $\varphi(a)$, $\varphi(b)$, $\varphi(c)$ be different atoms of B. Then there exists no chain $x > y > z$ such that $x, y, z \in \{a, a', b, b', c, c'\}$.

Proof. It is easy to see that, say, $x, y \in \{a, a'\}$ is impossible. Thus, without loss of generality we can assume $x \in \{a, a'\}$, $y \in \{b, b'\}$, $z \in \{c, c'\}$ and $x > y > z$. According to Lemma A and Lemma C we get

$$\varphi(\{x \wedge y', (x \wedge y')'\}) = \{\varphi(x) \vee \varphi(y), (\varphi(x) \vee \varphi(y))'\},$$
$$\varphi(\{x \wedge z', (x \wedge z')'\}) = \{\varphi(x) \vee \varphi(z), (\varphi(x) \vee \varphi(z))'\} \text{ and}$$
$$\varphi(\{y \wedge z', (y \wedge z')'\}) = \{\varphi(y) \vee \varphi(z), (\varphi(y) \vee \varphi(z))'\}.$$

Really, it is easy to check that $x \wedge y' \in \{0, 1, x, y, x', y'\}$ is impossible. Therefore, $\varphi(x \wedge y') \notin \{0, 1, \varphi(x), \varphi(y), (\varphi(x))', (\varphi(y))'\}$. Hence, by Lemma C and Theorem 1,

$$\varphi(\{x \wedge y', (x \wedge y')'\}) = \{\varphi(x) \vee \varphi(y), (\varphi(x) \vee \varphi(y))'\}.$$

Similarly one can establish the remaining two statements. It is not difficult to verify that

$$\varphi((y \wedge z') \vee x') \in \{\varphi(x) \vee \varphi(y) \vee \varphi(z), (\varphi(x) \vee \varphi(y) \vee \varphi(z))'\}.$$

Evidently, $(y \wedge z') \vee x' \notin \{0, 1, x, y, z, x', y', z'\}$. Since

$$(y \wedge z') \vee x' \notin \{x \wedge y', x \wedge z', y \wedge z', (x \wedge y')', (x \wedge z')', (y \wedge z')'\},$$

we see that

$$\varphi((y \wedge z') \vee x') \notin \{0, 1, \varphi(x), \varphi(y), \varphi(z), (\varphi(x))', (\varphi(y))', (\varphi(z))', \varphi(x) \vee \varphi(y),$$
$$\varphi(x) \vee \varphi(z), \varphi(y) \vee \varphi(z), \varphi(x) \vee \varphi(y))', (\varphi(y) \vee \varphi(z))', (\varphi(x) \vee \varphi(z))'\}.$$

Now, again by Lemma C

$$\varphi((y \wedge z') \vee x') \in \{\varphi(x) \vee \varphi(y) \vee \varphi(z), (\varphi(x) \vee \varphi(y) \vee \varphi(z))'\}.$$

Consider $y \wedge z'$. Clearly, $y \wedge z' = ((y \wedge z') \vee x') \wedge y$. It follows that

$$\varphi(y \wedge z') \in [\varphi((y \wedge z') \vee x'), \varphi(y)] = [\varphi(x) \vee \varphi(y) \vee \varphi(z), \varphi(y)] =$$
$$= \{\varphi(y), (\varphi(y))', \varphi(x) \vee \varphi(y) \vee \varphi(z), (\varphi(x) \vee \varphi(y) \vee \varphi(z))', \varphi(x) \vee \varphi(z),$$
$$(\varphi(x) \vee \varphi(z))', 0, 1\}, \text{ a contradiction with}$$
$$\varphi(\{y \wedge z', (y \wedge z')'\}) = \{\varphi(y) \vee \varphi(z), (\varphi(y) \vee \varphi(z))'\}.$$

Lemma 3. Let φ be an autogenomorphism on a Boolean algebra B with $|B| \geqslant 8$. Let $\varphi(a)$, $\varphi(b)$, $\varphi(c)$ be different atoms of B. Then

(i) $a < b$ implies $b > c$ or $b \vee c = 1$,
(ii) $a > b$ implies $b < c$ or $b \wedge c = 0$,
(iii) $a \wedge b = 0$ implies $b < c$ or $b \wedge c = 0$,
(iv) $a \vee b = 1$ implies $b > c$ or $b \vee c = 1$.

Proof. (i) It is easy to check that $b < c$ or $b \wedge c = 0$ imply $a < b < c$ or $a < b < c'$, which yields a contradiction with Lemma 2. By the same argument we show that $b > c$ or $b \vee c = 1$ is impossible in case (ii). Similarly, $b > c$ or $b \vee c = 1$ implies $a' > b > c$ or $a' > b > c'$ in (iii), which is impossible. Eventually, $b < c$ or $b \wedge c = 0$ implies $a' < b < c$ or $a' < b < c'$ in (iv), which again contradicts Lemma 2. An application of Lemma 1 concludes the proof.

Theorem 4. Let φ be an autogenomorphism on a Boolean algebra B with $|B| = 2^n$, $(n > 2)$. Then there exists a similar autogenomorphism ψ on B such that $a \wedge b = 0$ for arbitrary two atoms $\psi(a)$, $\psi(b)$ of B.

Proof. First we prove that for arbitrary three elements $a, b, c \in B$ such that $\varphi(a)$, $\varphi(b)$, $\varphi(c)$ are different atoms in B, there exists a similar autogenomorphism ψ such that $\psi(a_1)=\varphi(a)$, $\psi(b_1)=\varphi(b)$, $\psi(c_1)=\varphi(c)$ and $a_1 \wedge b_1 = a_1 \wedge c_1 = b_1 \wedge c_1 = 0$.

By Lemma 3 four cases can occur:

(i) $a_1 = a$, $b_1 = b'$, $c_1 = c$ or $a_1 = a$, $b_1 = b'$, $c_1 = c'$;

(ii) $a_1 = a'$, $b_1 = b$, $c_1 = c'$ or $a_1 = a'$, $b_1 = b$, $c_1 = c$;

(iii) $a_1 = a$, $b_1 = b$, $c_1 = c'$ or $a_1 = a$, $b_1 = b$, $c_1 = c$;

(iv) $a_1 = a'$, $b_1 = b'$, $c_1 = c$ or $a_1 = a'$, $b_1 = b'$, $c_1 = c'$.

In all the cases we have $a_1 \wedge b_1 = b_1 \wedge c_1 = 0$. A repeated application of Theorem 3 and Lemma B ensures the existence of a similar autogenomorphism ψ satisfying the above conditions. It remains only to show that $a_1 \wedge c_1 = 0$.

Again, by Lemma 3, we have to verify three cases: $a_1 < c_1$ or $a_1 > c_1$ or $a_1 \vee c_1 = 1$. But $a_1 < c_1$ and $b_1 \wedge c_1 = 0$ imply $a_1 < c_1 < b_1'$, a contradiction; similarly, $a_1 > c_1$ and $a_1 \wedge b_1 = 0$ imply $b_1' > a_1 > c_1$, which contradicts Lemma 2.

Eventually, $b_1 \wedge a_1 = b_1 \wedge c_1 = 0$ implies $0 = b_1 \wedge (a_1 \vee c_1) = b_1$ in the last case, which is a contradiction to Theorem 1. Hence $a_1 \wedge c_1 = 0$. Now it is easy to prove Theorem using the induction on the number of atoms of B.

Remark 3. Let B be a Boolean algebra with $|B| = 2^n$ (n is finite). The nonzero elements $a_1, \ldots, a_n \in B$ satisfying $a_i \wedge a_j = 0$ for $i \neq j$ are exactly all atoms in B. Thus, according to Theorem 4, the following statement is true.

Lemma 4. Let φ be an autogenomorphism on finite Boolean algebra B. Then φ restricted to the set of atoms can be written in the form $\varphi = h \circ \psi$, where ψ is a bijection on the set of all atoms of B and $h(e) \in \{e, e'\}$ for each atom e.

Suppose we have a Boolean algebra B with 2^n elements (n is finite). An element $a \in B$ is said to have a length k if $a = e_1 \vee \ldots \vee e_k$, where $e_1, \ldots, e_k$ are different atoms of B. Notation $l(a) = k$. Now we can formulate

Theorem 5. Let φ be an autogenomorphism on a finite Boolean algebra with 2^n elements. Let $l(a) = k$, $0 \leqslant k \leqslant n$, for $a \in B$. Then $l(\varphi(a)) \in \{k, n-k\}$.

Proof. By induction on the length. For $k = 1$ it is true by Lemma 4. Suppose that $\frac{n}{2} \geqslant k > 1$ and that the statement is true for all elements a with $l(a) < k$. By hypothesis $a = e_1 \vee \ldots \vee e_k$. Therefore, $\varphi(a) \in [\varphi(e_1 \vee \ldots \vee e_{k-1}), \varphi(e_k)] = K$.

Denote $\varphi(e_1 \vee \ldots \vee e_{k-1})$ by b. First we show that $|K| \leqslant 8$. $\varphi(e_k)$ or $(\varphi e_k))'$ is an atom of B and b or b' has the length $k-1$. Without loss of generality we can assume that $\varphi(e_k)$ is an atom. Two cases can arise: $b \wedge \varphi(e_k) = 0$ or $b \geqslant \varphi(e_k)$, i.e. $b' \wedge \varphi(e_k) = 0$. By Lemma D, $K = \{0, 1, \varphi(e_k), (\varphi(e_k))', b, b', c, c'\}$, where $c = b \vee \varphi(e_k)$ in the first case and $c = b \wedge (\varphi(e_k))'$ in the second one. Now, let L_i denote the set of elements of length i and set $M_i = \bigcup_{j=1}^{i} (L_j \cup L_{n-j})$.

So by induction hypothesis $\varphi(M_i) = M_i$ for $i = 1, \ldots, k-1$. (It is a bijection of finite sets.) Since $a \notin M_{k-1}$, the length of $\varphi(a)$ is greater than $k-1$ and smaller than $n-k+1$. Therefore, $\varphi(a) \in \{c, c'\}$. Assume $l(b)=k-1$. We claim that $b \wedge \varphi(e_k)=0$. Suppose to the contrary that $b \geqslant \varphi(e_k)$. Hence, $c=b \wedge (\varphi(e_k))'$, $c'=b' \vee \varphi(e_k) \in M_{k-1}$, which is a contradiction. Thus, $b \wedge \varphi(e_k)=0$, as claimed, and consequently $\varphi(a) \in \{b \vee \varphi(e_k),\ b' \wedge (\varphi(e_k))'\}$. By the same reasoning we obtain $\varphi(a) \in \{b' \vee \varphi(e_k),\ b \wedge (\varphi(e_k))'\}$ for $l(b)=n-k+1$. The proof is finished.

Theorem 6. Let φ be a bijection on a finite Boolean algebra B. Then φ is an autogenomorphism on B if and only if $\varphi = h \circ \psi$, where ψ is an automorphism on B and h is a bijection satisfying $h(a) \in \{a, a'\}$ for every $a \in B$.

Proof. Let φ be an autogenomorphism on B with $|B|=2^n$. Suppose that $\{e_1, \dots, e_n\}$ is the set of all atoms of B. First we prove that there exists an autogenomorphism φ_1 on B similar to φ such that

$$(*) \qquad \varphi_1(a)=\varphi_1(e_1) \vee \dots \vee \varphi_1(e_k) \text{ whenever } a=e_1 \vee \dots \vee e_k .$$

In accordance with Theorem 5 there exists an autogenomorphism ψ_1 on B satisfying

$$\psi_1(a)=\begin{cases} \varphi(a), \text{ if } l(\varphi(a))=l(a) \\ (\varphi(a))', \text{ if } l(\varphi(a))=n-l(a). \end{cases}$$

We claim that ψ_1 satisfies (*) for all $a \in B$ with $l(a)<\frac{n}{2}$ or $l(a)>\frac{n}{2}$. Really, let $l(a)<\frac{n}{2}$. We shall proceed by induction on $l(a)$. For $a=0$ is $\psi_1(a)=0$. Similarly, if $l(a)=1$, i.e. a is an atom, then $\psi_1(a)$ is also an atom. Thus, for $0 \leqslant l(a) \leqslant 1$, ψ_1 satisfies property (*). Suppose that $1 \leqslant l(a)=k<\frac{n}{2}$ and that ψ_1 satisfies (*) for all $b \in B$ with $0 \leqslant l(b)<k$. Assume $a=e_1 \vee \dots \vee e_{k-1} \vee e_k$. Set $b=e_1 \vee \dots \vee e_{k-1}$. By induction hypothesis is $\psi_1(e_k)$ an atom of B and $\psi_1(b)=\psi_1(e_1) \vee \dots \vee \psi_1(e_{k-1})$.

Two cases can occur: $\psi_1(b) \geqslant \psi_1(e_k)$ or $\psi_1(b) \wedge \psi_1(e_k)=0$.

The first case implies $\psi_1(e_k)=\psi_1(e_i)$ for some $1 \leqslant i \leqslant k-1$, which is impossible, as ψ_1 is a bijection. Now, in the second case, $\psi_1(a) \in [\psi_1(b), \psi_1(e_k)]$. By Lemma D, $\psi_1(a)=\psi_1(b) \vee \psi_1(e_k)$, as $\psi_1(b) \vee \psi_1(e_k)$ is the only element from $[\psi_1(b), \psi_1(e_k)]$ with length equal to k.

Therefore, ψ_1 satisfies (*) for all elements $a \in B$ with $l(a)<\frac{n}{2}$. Take $a \in B$ with $l(a)>\frac{n}{2}$. Evidently, $l(a')<\frac{n}{2}$ and $\psi_1(a')=\psi_1(e_1) \vee \dots \vee \psi_1(e_k)$, whenever $a'=e_1 \vee \dots \vee e_k$. Clearly, $a=e_{k+1} \vee \dots \vee e_n$. Since $\psi_1(a')=(\psi_1(a))'$ (see Theorem 1), then $\psi_1(a)=\psi_1(e_{k+1}) \vee \dots \vee \psi_1(e_n)$.

Now, let $a \in B$ with $l(a)=\frac{n}{2}$ for n even. Clearly, $\psi_1(a)=\varphi(a)$ or $\psi_1(a)=$

$(\varphi(a))'$. Let $a = e_1 \vee \ldots \vee e_k$, $k = \frac{n}{2}$. It is easy to show that $\{\psi_1(a), (\psi_1(a))'\} = \{\psi_1(e_1) \vee \ldots \vee \psi_1(e_k), (\psi_1(e_1) \vee \ldots \vee \psi_1(e_k))'\}$. Really, set $b = e_1 \vee \ldots \vee e_{k-1}$. Thus, $\psi_1(b) = \psi_1(e_1) \vee \ldots \vee \psi(e_{k-1})$. Clearly, $\psi_1(a) \in [\psi_1(b), \psi_1(e_k)]$. But the only two elements of this subalgebra with length equal to k, are exactly $\psi_1(e_1) \vee \ldots \vee \psi_1(e_k)$ and $(\psi_1(e_1) \vee \ldots \vee \psi_1(e_k))'$, hence $\{\psi_1(a), (\psi_1(a))'\} = \{\psi_1(e_1) \vee \ldots \vee \psi_1(e_k), (\psi_1(e_1) \vee \ldots \vee \psi_1(e_k))'\}$. Define a new autogenomorphism ψ on B as follows:

$$\psi(b) = \begin{cases} \psi_1(b) \text{ for } l(b) = k \neq \frac{n}{2} \\ \\ \psi_1(e_1) \vee \ldots \vee \psi_1(e_k), \text{ whenever } b = e_1 \vee \ldots \vee e_k,\ k = \frac{n}{2}. \end{cases}$$

Clearly, ψ is similar to ψ_1 and ψ satisfies property (*) for each $a \in B$. Now, we claim that ψ is exactly a mapping ψ from the Theorem.

Evidently, $\{\psi(a), (\psi(a))'\} = \{\psi_1(a), (\psi_1(a))'\} = \{\varphi(a), (\varphi(a))'\}$ for each $a \in B$, hence we can express φ in the form $\varphi = h \circ \psi$, where h is a bijection on B satisfying $h(a) \in \{a, a'\}$.

Now, to complete the first part of the proof, it remains to show that ψ is an automorphism on B.

Let $a, b \in B$. We show that $\psi(a \wedge b) = \psi(a) \wedge \psi(b)$.

We can express a, b in the form:

$$a = e_1 \vee \ldots \vee e_k \vee e_{k+1} \vee \ldots \vee e_{k+l},\ b = e_1 \vee \ldots \vee e_k \vee e_{k+l+1} \vee \ldots \vee e_{k+l+m}$$

with

$$\{e_{k+1}, \ldots, e_{k+l}\} \cap \{e_{k+l+1}, \ldots, e_{k+l+m}\} = \emptyset.$$

Then

$$\psi(a) = \psi(e_1) \vee \ldots \vee \psi(e_k) \vee \psi(e_{k+1}) \vee \ldots \vee \psi(e_{k+l}),$$

$$\psi(b) = \psi(e_1) \vee \ldots \vee \psi(e_k) \vee \psi(e_{k+l+1}) \vee \ldots \vee \psi(e_{k+l+m}),$$

and

$$\psi(a \wedge b) = \psi(e_1 \vee \ldots \vee e_k) = \psi(e_1) \vee \ldots \vee (e_k) = \psi(a) \wedge \psi(b).$$

Since $\psi(a') = (\psi(a))'$ for each $a \in B$, we have $\psi(a \vee b) = \psi(a) \vee \psi(b)$ for $a, b \in B$, hence ψ is an automorphism.

Conversely, let $\varphi = h \circ \psi$, where ψ is an automorphism on B and h is a bijection satisfying $h(a) \in \{a, a'\}$ for every $a \in B$. Clearly, both mappings h and ψ are autogenomorphisms on B and by Lemma B φ is an autogenomorphism, too.

The proof is finished.

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SÚHRN

AUTOGENOMORFIZMY NA KONEČNÝCH BOOLOVSKÝCH ALGEBRÁCH

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Genomorfizmus je také zovšeobecnenie homorfizmu, ktoré možno uvažovať medzi algebrami rôznych typov. Každý autogenomorfizmus φ na konečnej boolovskej algebre B má tvar $\varphi = h \circ \psi$, kde ψ je automorfizmus a $h(a) \in \{a, a'\}$ pre každé $a \in B$.

РЕЗЮМЕ

АВТОГЕНОМОРФИЗМЫ НА КОНЕЧНЫХ БУЛЕВЫХ АЛГЕБРАХ

Э. Котлебова, Братислава

Геноморфизм такое обобщение гомоморфизма, которое можно изучать между алгебрами разных типов. Каждый геноморфизм φ на конечной Булевой алгебре B имеет форму $\varphi = h \circ \psi$, где ψ – автоморфизм и $h(a) \in \{a, a'\}$ для всех $a \in B$.