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NON-ORIENTABILITY OF OPEN SUBMANIFOLDS

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1. Local decomposition of manifolds

There is a well-known result on orientability of open submanifolds, see e.g. [3; Prop. 22.7]:

Theorem A. (a) An open submanifold of an orientable manifold[1]) is orientable.

(b) A manifold is orientable if and only if all its components are orientable.

The purpose of the present paper is to prove that the part (a) of Theorem A is convertible under some additional assumption:

Theorem 1. Let M' be an open submanifold of a manifold M such that the complement $M - M'$ of M' in M does not locally decompose M. Then M' is orientable if and only if M is orientable.

Recall that a subset A of a topological space X does not locally decompose X if there exists a basis $\mathcal{B}$ of the topology of X such that both sets U and $U - A$ are connected and non-empty for every $U \in \mathcal{B}$. Note that the connectivity of U is a consequence of the connectivity of $U - A$.

The crucial step of the proof of Theorem 1 is contained in the following

Proposition 1. Let A be a subset of a connected topological space X such that the complement $X - A$ does not locally decompose X. Then A is connected.

Theorem 1 and Proposition 1 will be proved in the next section. In order to obtain a simple sufficient condition for a subset A of a manifold M which does not locally decompose M, let us turn our attention to the classical result of Menger and Urysohn, see e.g. [1; Chap. 8., Th. 2][2])

[1]) By a manifold we mean a topological manifold, i.e. a locally euclidean Hausdorff space not necessarily connected or separable.

[2]) Under dimension dim X of a topological space X we mean the Lebesgue's „plaster" dimension of X.

Theorem B. Let A be a subset of a connected separable m-dimensional manifold M. If dim $A < m-1$ then A does not decompose M, i. e. the set $M - A$ is connected.

Remark 1. Sitnikov [5] has constructed a 2-dimensional subset N of R^3 which does not locally decompose R_3. Therefore the condition dim $A < m-1$ is not necessary for the conclusion of Theorem B.

Let A be a subset of a separable m-dimensional manifold M such that dim $A < m-1$. Then every domain U in M is a connected separable m-dimensional manifold and dim $U \cap A < m-1$ because of the monotony of dimension in separable manifolds. The set $\mathcal{D}$ of all domains in M is a basis of the topology of M, thus, under the assumptions of Theorem B, the set A does not locally decompose M. As a consequence of Theorem 1 we obtain

Corollary 1. Let M' be an open submanifold of a separable m-dimensional manifold M such that dim $(M - M') < m-1$. Then M' is orientable if and only if M is orientable.

Remark 2. The assumption on submanifold M' in Theorem 1 is not necessary for the conclusion of Theorem even if M' is connected and dense in M as follows from the example: M is the real projective plane RP^2, A_1 a projective line in RP^2 and A_2 a proper closed segment on the line A_1. Both open submanifolds $M_1 = M - A_1$ and $M_2 = M - A_2$ do not satisfy assumption in Theorem 1. The first one is orientable and the second one is non-orientable.

If we omit the assumption on separability of M in Theorem B, then the various definitions of dimension need not to coincide and Theorem B would became questionable. Nevertheless, we can define dimension of a subset N of a manifold M at a point $p \in M$ (M not necessarily connected or separable), denoted by $\dim_p N$, as the minimum of dim $U \cap N$, where U is a neighbourhood of p in M homeomorphic to an open subset of an euclidean space R^m. Corollary 1 can be strengthened in the following way:

Corollary 2. Let M' be an open submanifold of a manifold M such that $\dim_p (M - M') < \dim_p M - 1$ for every $p \in M$. Then M' is orientable if and only if M is orientable.

Proof. To every point $p \in M$ let us choose a neighbourhood U_p homeomorphic to an open connected subset of some euclidean space R^m such that $\dim_p (M - M') = \dim (U_p - M')$ and $\dim_p M = \dim U_p$. Then both $A = U_p - M'$ and $M = U_p$ satisfy the assumptions of Theorem B. Hence, as above, $U_p - M'$ does not locally decompose U_p. Therefore $M - M'$ does not locally decompose M, so we can make use of Theorem 1.

Combining the considerations in the proof of Corollary 2 with Proposition 1, we get the following generalization of Theorem B:

Corollary 3. Let A be a subset of a connected manifold M such that $\dim_p A < \dim_p M - 1$ for every $p \in M$. Then A does not decompose M.

Remark 3. It is clear that $\dim_p N = -1$ for every point p not belonging to the closure $\bar{N}$ of a subset N of a manifold M. Hence, to fulfill the essumptions in Corellaries 2 and 3, it sufficies to verify the condition $\dim_p (M - M') < \dim_p M - 1$ or $\dim_p A < \dim_p M - 1$ only for $p \in M - M'$ or $p \in \bar{A}$, respectively.

Remark 4. If the manifold M is separable then every subset P of M is paracompact. A result of Dowker and Nagami on local dimensions in paracompact spaces, c.f. [1; Th. 4.21], says that $\dim P = \sup \{\dim_p P; p \in M\}$. It means that if M is separable, the assumption of Theorem B and those of Corollary 3 are equivalent. Similarly, the assumptions of Corollaries 1 and 2 are equivalent provided M separable.

2. Proof of Pròposition 1 and Theorem 1

Proof of Proposition 1. Let C be a component of A. According to our assumptions, there exists a basis $\mathcal{B}$ of the topology of X such that both sets U and $U - (X - A) = U \cap A$ are connected and non-empty for every $U \in \mathcal{B}$. Let $\mathcal{B}_1$ be a subset of $\mathcal{B}$ consisting of all $U \in \mathcal{B}$ for which $U \cap A \subset C$. The space X is a union of its two open subsets $\bigcup \mathcal{B}_1$ and $\bigcup(\mathcal{B} - \mathcal{B}_1)$. It is not difficult to show that the set $\mathcal{B} - \mathcal{B}_1$ consists of all $U \in \mathcal{B}$ for which $U \cap A \subset A - C$. Therefore $U_1 \cap U_2 = \emptyset$ for every $U_1 \in \mathcal{B}_1$, $U_2 \in \mathcal{B}_2$. If not, then there exists an element U of $\mathcal{B}$ such that $U \subset U_1 \cap U_2$. For such U we have $U \cap A \subset U_1 \cap U_2 \cap A = (U_1 \cap A) \cap (U_2 \cap A) \subset C \cap (A - C) = \emptyset$, thus $U \cap A = \emptyset$ which is a contradiction. We have just shown that the sets $\bigcup \mathcal{B}_1$ and $\bigcup(\mathcal{B} - \mathcal{B}_1)$ are disjoint. The set $\bigcup \mathcal{B}_1$ is non-empty because it contains the non-empty set C. The connectivity of X implies now $\bigcup(\mathcal{B} - \mathcal{B}_1) = \emptyset$, therefore $\mathcal{B} - \mathcal{B}_1 = \emptyset$, thus $A - C = \emptyset$ and so $A = C$. Proposition 1 is proved.

Before starting the proof of Theorem 1 let us recall that a connected manifold N is non-orientable if and only if the total space $O(N)$ of its orientation covering $\omega_N = (O(N), \pi_N, N)$ is connected, see e.g. [3; VIII. 2.11]. The manifold $O(N)$ consists of all generators α_p of all local homology groups $H_n(N, N - \{p\})$, $p \in N$ where $n = \dim N$. The projection π_N: $O(N) \to N$ is defined by $\pi_N(\alpha_p) = p$ for every $p \in N$. The fibration ω_N is a double covering because $H_n(N, N - \{p\}) \cong \mathbf{Z}$ for every $p \in N$. Using excision we can show easily that $O(M')$ is homeomorphic to the subspace $\pi_M^{-1}(M')$ of $O(M)$ for every open submanifold M' of a manifold M.

Proof of Theorem 1. Owing to Theorem A it suffices to prove the following assertion: If M is connected and non-orientable, then M' is non-orientable. The assumption on M' implies that $\pi_M^{-1}(M - M')$ does not locally decompose $O(M)$ because π_M is a surjective local homeomorphism. An immediate application of Proposition 1 concludes now the proof of Theorem 1.

3. Applications

1. Let m, n be integers such that $0<m<n$. Let us consider the Grassmann manifold $G_m(RP^n)$ or $G_m(E^n)$ consisting of all m-dimensional planes in the n-dimensional real projective space RP^n or in the n-dimensional real auclidean space E^n, respectively. The manifold $G_m(E^n)$ can be regarded as an open submanifold of $G_m(RP^n)$:

$$G_m(E^n)=G_m(RP^n)-G_m(RP^{n-1}),$$

where RP^{n-1} is a fixed hyperplane in RP^n. We have

$$\dim G_m(RP^n)-\dim G_m(RP^{n-1})=(m+1)(n-m)-(m+1)(n-m-1)=m+1>1$$

what means that the assumptions of Corollary 1 are fulfilled. It is well-known that $G_m(RP^n)$ is orientable if and only if n is odd. So we have proved

Theorem 2. The manifold $G_m(E^n)$, $0<m<n$, is orientable if and only if n is odd.

2. Let $\xi=(E, p, S^n)$ be a fibre bundle[3]) over the n-dimensional sphere S^n with the fibre F and let $n>1$. Let us choose a point $x_0\in S^n$. Then $\dim E-\dim p^{-1}(x_0)=n$, hence $\dim p^{-1}(x_0)<\dim E-1$. According to Corollary 1 E is orientable if and only if $E-p^{-1}(x_0)$ is orientable. But $E-p^{-1}(x_0)$ is the total space of the restricted fibre bundle $\xi\,|\,(S^n-\{x_0\})$ which is a trivial fibration because its base space $S^n-\{x_0\}$ is homeomorphic to R^n and R^n is contractible. Therefore $E-p^{-1}(x_0)$ is homeomorphic to the product-manifold $\mathrm{R}^n\times F$ which is orientable if and only if F is orientable. Thus we have proved the following

Theorem 3. The total space of a fibre bundle ξ over S^n, $n>1$ is orientable if and only if the fibre of ξ is orientable.

3. Let $\xi=(E, p, B)$ be a vector bundle with the fibre F, g a riemannian metric on ξ and $o\colon B\to E$ the zero cross section of ξ. We have $\dim E-\dim o(B)=\dim F$. If $\dim F>1$, then Corollary 1 implies that E is orientable if and only if $E-o(B)$ is orientable. On the other hand, $E-o(B)$ is naturally homeomorphic to $SE\times\mathrm{R}$, where $SE=\{x\in E;\ g(x, x)=1\}$ is the total space of the sphere bundle $S\xi$. Thus we have proved the following

Theorem 4. The total space of a vector bundle with the fibre dimension $n>1$ admitting a riemannian metric is an orientable manifold if and only if the total space of the corresponding sphere bundle is an orientable manifold.

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[3]) Under a fibre bundle we mean a fibration associated with a locally trivial principal fibration in sense of [4]. We shall assume that all fibrations belong to the category of topological manifolds and continous maps.

Remark 5. The assumption of Theorem 4 on the existence of a riemannian metric is fulfilled if the base space is a separable manifold.

4. As an illustration of the above Theorems 3 and 4 let us consider the total space E_k^n of the canonical vector bundle γ_k^n over $G_k(\mathrm{R}^n)$, $n > k \geqq 1$. Recall that E_k^n is the submanifold of $G_k(\mathrm{R}^n) \times \mathrm{R}^n$ consisting of all couples $(x, v) \in G_k(\mathrm{R}^n) \times \mathrm{R}^n$ for which $v \in x$.

Theorem 5. The total space E_k^n of the canonical vector bundle γ_k^n over $G_k(\mathrm{R}^n)$ is orientable if and only if n is odd.

Proof. The case $n = 1$ is trivial. First assume $n \geqq 2$ ank $k > 1$. Let SE_k^n be the total space of the sphere bundle corresponding to γ_k^n with respect to the riemannian metric on γ_k^n induced by the standart inner product in R^n. Then SE_k^n can be regarded as the total space of a fibre bundle over S^{n-1} given by the projection $(x, v) \mapsto v$ for all $(x, v) \in SE_k^n$. The fibre of this fibre bundle is $G_{k-1}(\mathrm{R}^{n-1})$. Using the known result on orientability of Grassmann manifolds and Theorems 3 and 4 we see that Theorem 5 holds for all n, $k \geqq 2$.

Now, let $n \geqq 2$ and $k = 1$. Let us consider a map $f\colon \mathrm{R}^n - B^n \to E_1^n$ defined by $f(v) = (\langle v \rangle, (\|v\| - 1)v)$ for all $v \in \mathrm{R}^n - B^n$, where B^n is the standart unite open ball in R^n and $\langle v \rangle$ is the vector-subspace of R^n spanned by the vector v. It is not difficult to show that the map f induces a homeomorsphism of the factor space $(\mathrm{R}^n - B^n)/f$ with E_1^n. However, $(\mathrm{R}^n - B^n)/f$ is homeomorphic to the open submanifold $RP^n - \{\text{point}\}$ of the real projective space RP^n and, by Corollary 1, $RP^n - \{\text{point}\}$ is orientable if and only if RP^n is orientable, i.e. if and only if n is odd.

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SÚHRN

NEORIENTOVATEĽNOSŤ OTVORENÝCH PODVARIET

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V práci sa dokazuje, že ak doplnok $M - M'$ otvorenej podvariety M' topologickej variety M lokálne nerozkladá varientu M, tak varieta M' je orientovateľná práve vtedy, keď je orientovateľná varieta M. Pomocou tohoto výsledku sa dokazujú nutné a postačujúce podmienky orientovateľnosti niektorých variet. Okrem iného sa ukazuje, že totálny priestor fibrovaného priestoru ξ nad sférou S^n, $n \geqq 2$, je orientovateľnou varietou práve vtedy, keď je orientovateľnou varietou fiber fibrovaného priestoru ξ.

РЕЗЮМЕ

НЕОРИЕНТИРУЕМОСТЬ ОТКРЫТЫХ ПОДМНОГОБРАЗИЙ

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В работе доказывается, что открытое подмногообразие M' многообразия M не обязательно связного или сепарабельного, дополнение которого локально не разрезает M, ориентируемо тогда и только тогда, когда ориентируемо многообразие M. В качестве приложения решена проблема ориентируемости некоторых многообразий связаных с многообразиями Грассманна и с расслоеными пространствами.