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ELEMENTARY NONSTANDARD APPROACH TO METRIC SPACES

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Introduction

The nonstandard methods of the mathematical analysis are frequently used also in the topology (see e.g. [2], [8]). Of course the methods of the mathematical logic are used in a large extent. The aim of this paper is to present an elementary nonstandard theory of metric spaces without deep methods of the mathematical logic.

The paper is more or less of a methodical character. We construct at first for a given set X a set $*X$. The last serves as a basic set for the nonstandard considerations. In case of $X=R$ the set $*R$ is a basic set for the construction of hyperreal numbers. Having such a set $*X$ for a metric space (X, ϱ) we are able to introduce, using the metric ϱ, the nonstandard notions. Then we formulate in the nonstandard language some necessary and sufficient conditions for sets to be compact, closed, open etc. Nonstandard formulation of the convergence and some related notions are given as well as nonstandard proofs of some well known theorems in the theory of metric spaces. It is shown also that on the extended set $*X$ an extended metric $*\varrho$ may be defined. At last product spaces are studied and the notion of continuity in the nonstandard formulation is discussed.

We do not give the definitions of the fundamental notions in metric spaces. The fundamental theorems from standard theory of metric spaces are also supposed to be known to the reader.

1. Extension of a set by means of an ultrafilter

An equivalence on the sets of all sequences of points of a given set X will be defined. To define the equivalence an ultrafilter will be used. The collection of all the equivalence classes will be the required extension of the set X.

Definition 1.1. Let N be the set of all positive integers. By a nontrivial ultrafilter of subsets of N we mean a collection $\mathcal{F}$ of subsets of X such that

(1) $\emptyset \notin \mathcal{F}$,
(2) $E, F \in \mathcal{F} \Rightarrow E \cap F \in \mathcal{F}$,
(3) $E \in \mathcal{F}, E \subset F \subset N \Rightarrow F \in \mathcal{F}$,
(4) $E \subset N \Rightarrow$ either $E \in \mathcal{F}$ or $N - E \in \mathcal{F}$,
(5) $\mathcal{F} \supset \mathcal{G}\{=\{E \subset N\ N - E$ is finite$\}$.

Remark 1.1. The existence of such $\mathcal{F}$ follows from Zorn's Lemma (see e.g. [5]). Throughout all the paper $\mathcal{F}$ will be fixed.

Definition 1.2. Let $X \neq \emptyset$ be any set and $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ sequences of points of X. The sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ will be called equivalent with respect to $\mathcal{F}$ (notation $(a_n)_{n=1}^{\infty} \sim (b_n)_{n=1}^{\infty}$ if there exists $F \in \mathcal{F}$ such that $a_k = b_k$ for any $k \in F$.

Definition 1.3. Let $X \neq \emptyset$ be arbitrary and let $\tilde{X}$ be the set of all sequences of points of X. The set $*X$ of all the equivalence classes with respect to the equivalence which was introduced above will be called the extension of X (with respect to $\mathcal{F}$).

Remark 1.2. a) The set X may be embedded into $*X$ in the following way. We identify any point $a \in X$ with the class $\bar{a}$ into which the sequence $(a, a, \dots)$ belongs. Evidently $\bar{a} \neq \bar{b}$ if $a \neq b$. In accordance with this identification we write $X \subset *X$ and sometime a instead of $\bar{a}$.

b) If X is an infinite set then $*X - X \neq \emptyset$, because the sequence $(a_1, a_2, a_3, \dots)$ of mutually distinct points of X is not equivalent to any of the sequences $(a, a, a, \dots)$, where $a \in X$. Thus if $\alpha \in *X$ is such that $(a_1, a_2, a_3, \dots) \in \alpha$ we have $\alpha \neq \bar{a}$ for any $a \in X$.

c) If $(a_n)_{n=1}^{\infty}$ is a sequence of points of X then the unique class into which $(a_n)_{n=1}^{\infty}$ belongs will be sometime denoted $[(a_n)_{n=1}^{\infty}]$ or shortly $[(a_n)]$.

Definition 1.4. If $A \subset X$ then the extension $*A$ of the set A is defined as follows

$$*A = \{\alpha \in *X \mid (a_n)_{n=1}^{\infty} \in \alpha \Rightarrow \exists F \in \mathcal{F}\ \forall k \in F\colon a_k \in A\}$$

The definition of $*A$ does not depend on the choice of $(a_n)_{n=1}^{\infty} \in \alpha$. In fact, if $(b_n)_{n=1}^{\infty} \in \alpha$ is any sequence, then $b_k = a_k$ for $k \in E$ where E is some set belonging to $\mathcal{F}$. Thus $b_k = a_k$ for any $k \in E \cap F \in \mathcal{F}$.

The following properties are direct consequences of Definition 1.4 and of the properties of $\mathcal{F}$.

Theorem 1.1. Let $X \neq \emptyset$; $A, B \subset X$. Then the following is true:
(a) $*\emptyset = \emptyset$; $A \subset *A$ in the sense of the "embedding" of X into $*X$,
(b) $A \subset B \Rightarrow *A \subset *B$,
(c) $*(A \cap B) = *A \cap *B$,
(d) $*(A \cup B) = *A \cup *B$,
(e) $*(A - B) = *A - *B$,

(f) $^*\{a\}=\bar{a}$ for any $a \in X$,

(g) The extensions in the sense of definitions 1.3 and 1.4 coincide.

Remark 1.3. If R is the set of all real numbers then the extension $^*R$ with suitable defined sum product and ordering is the set of all hyperreal numbers. The sum and the product is defined as follows: If $\alpha, \beta \in {}^*R$, $\alpha=[(a_n)_{n=1}^{\infty}]$, $\beta=[(b_n)_{n=1}^{\infty}]$ where $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ are arbitrarily chosen then $\alpha+\beta=[(a_n+b_n)_{n=1}^{\infty}]$, $\alpha\beta=[(a_nb_n)_{n=1}^{\infty}]$. If there is $F \in \mathscr{F}$ such that $a_k<b_k$ for $k \in F$, then we write $\alpha<\beta$. It can be easily seen that the sum, product and the ordering are unambigously defined. Futher one can see that $^*R$ with the above operations and ordering is an ordered field. The zero element is the class $\bar{0}$ containing the sequence $(0, 0, 0, \ldots)$ while the class $\bar{1}$ containing the sequence $(1, 1, 1, \ldots)$ is the unit of the field. The field is not Archimedean because e.g. for the class α containing $(1, 2, 3, \ldots)$ we have $\alpha>\bar{n}=n$ for any $n \in N$. Thus $0<\frac{1}{\alpha}<\frac{1}{n}$ for any $n \in N$. So we see that $^*R$ contains infinitesimal elements.

2. Metric space. Monad of a point

It will be perhaps usefull to explain the relation of our approach to that one given in systematic theories on nonstandard analysis. (Compare e.g. [2], [8]). Roughly speaking we are showing that to obtain the basic results in theory of metric spaces one need not develop the complete nonstandard theory with the "complete" superstructure (see e.g. [8] pp. 43—46). It is not necessary to go "too far" from the basic set on which usually the superstructure is constructed. Since we restrict to relatively simple case, deep principles of nonstandard analysis are omitted because the results are obtained in a direct way.

Define the relation $\approx$ which will mean that $\alpha, \beta \in {}^*X$ are infinitely near.

Definition 2.1. Let (X, ϱ) be a metric space, $^*X$ the extension of X. The elements $\alpha, \beta \in {}^*X$ will said to be infinitely near (notation $\alpha \approx \beta$) if for $(p_n)_{n=1} \in \alpha$, $(q_n)_{n=1} \in \beta$ and any $\varepsilon>0$ (ε real) there exists $F \in \mathscr{F}$ such that $\varrho(p_n, q_n)<\varepsilon$ for any $n \in F$.

Definition 2.2. If $p \in X$ then the set $\mu(p)$ of all $q \in {}^*X$ which are infinitely near to p will be called the monad of p. So $\mu(p)=\{q \in {}^*X \mid q \approx p\}$.

Theorem 2.1. Let (X, ϱ) be a metric space, $p, q \in X$. If $p \neq q$ then $\mu(p) \cap \mu(q)=\emptyset$.

Proof. Suppose $\mu(p) \cap \mu(q) \neq \emptyset$. Let $y \in \mu(p) \cap \mu(q)$ and $(a_n)_{n=1}^{\infty} \in y$. Then we have for any real number $\varepsilon>0$.

$\varrho(a_n, p)<\varepsilon$ for n belonging to a set $E \in \mathscr{F}$ and simultaneously

$\varrho(a_n, q)<\varepsilon$ for n belonging to a set $F \in \mathscr{F}$. Hence for $k \in E \cap F$

$$\varrho(p, q) \leqq \varrho(a_k, p)+\varrho(a_k, q)<2\varepsilon$$

So $\varrho(p, q)=0$. Thus $p=q$.

The notion of the monad of a point enables to characterize the important topological notions as are open, closed, compact sets etc.

Theorem 2.2. Let (X, ϱ) be a metric space, $x_0 \in X$. Then $\mu(x_0)=\bigcap_{n=1}^{\infty} {}^*O_{\frac{1}{n}}(x_0)=\bigcap_{\varepsilon>0} {}^*O_{\varepsilon}(x_0)$ where for a real positive δ $O_\delta(x_0)=\{y \in X \mid \varrho(x_0, y)<\delta\}$

Proof. a) If $y \in \mu(x_0)$ than for any $(a_n)_{n=1}^{\infty} \in y$ and any $\varepsilon>0$ we have $\varrho(a_n, x_0)<\varepsilon$ on a set $F \in \mathscr{F}$, i.e. $y \in {}^*O_\varepsilon(x_0)$. Thus $\mu(x_0) \subset {}^*O_\varepsilon(x_0)$ for any $\varepsilon>0$, hence $\mu(x_0) \subset \bigcap_{\varepsilon>0} O^*(x_0)$. The inclusion $\bigcap_{\varepsilon>0} {}^*O(x_0) \subset \mu(x_0)$ may be proved similarly

b) The equality $\bigcap_{n=1}^{\infty} {}^*O_{\frac{1}{n}}(x_0)=\bigcap_{\varepsilon>0} {}^*0_\varepsilon(x_0)$ is evident from the monotonicity of the * operation.

Theorem 2.3. Let (X, ϱ) be a metric space. A set $G \subset X$ is open if and only if $\mu(x_0) \in {}^*G$ for every point $x_0 \in G$.

Proof. a) If G is open, $x_0 \in G$, then there exists $\varepsilon_0>0$ such that $O_{\varepsilon_0}(x_0)<G$. Hence ${}^*O_{\varepsilon_0}(x_0) \subset {}^*G$. Thus

$$\mu(x_0)=\bigcap_{\varepsilon>0} {}^*O_\varepsilon(x_0) \subset {}^*O_{\varepsilon_0}(x_0) \subset {}^*G$$

b) Let G be not open. Then there exists $x_0 \in G$ such that $O_\varepsilon(x_0) \not\subset G$ for any $\varepsilon>0$. Thus for any n there is $y_n \in O_{\frac{1}{n}}(x_0)$ with $y_n \notin {}^*G$. Put $F_n=\{k \in N \mid k>n\}$. Then $F_n \in \mathscr{F}$. For any $k \in F_n$ we have $y_k \in O_{\frac{1}{k}}(x_0)$ hence

$$\varrho(y_k, x_0)<\frac{1}{k}<\frac{1}{n}$$

Thus $y=[(y_n)] \in \mu(x_0)$. But it follows from the construction y that $y \notin {}^*G$. Hence $\mu(x_0) \not\subset {}^*G$.

Theorem 2.4. Let (X, ϱ) be a metric space. A set $F \subset X$ is closed if and only if for every $p \in X$

$$\mu(p) \cap {}^*F \neq \emptyset \Rightarrow p \in F.$$

Proof. F is closed if and only if $X-F$ is open. But $X-F$ is open if and only if for any $p \in X-F$ we have $\mu(p) \subset {}^*(X-F)={}^*X-{}^*F$. But the last is equivalent to the implication $p \in X-F \Rightarrow \mu(p) \cap {}^*F=\emptyset$.

3. The convergence in a nonstandard formulation

In a nonstandard form the convergence of a sequence may be described as follows.

Theorem 3.1. A sequence $(p_n)_{n=1}^{\infty}$ of points of a metric space (X, ϱ) converges to a point $p \in X$ if and only if $\mu(p)$ is the unique monad containing all $[(p_{k_n})]$ where $(p_{k_n})_{n=1}^{\infty}$ is any subsequence of $(p_n)_{n=1}^{\infty}$.

Proof. Take any $(p_n)_{n=1}^{\infty}$ converging to p. Let $(p_{k_n})_{n=1}^{\infty}$ be any subsequence of $(p_n)_{n=1}^{\infty}$. For any $\varepsilon>0$ there exists n_0 such that $\varrho(p_{k_n}, p)<\varepsilon$ whenever $n \geqq n_0$. Since the set $F=\{n \mid n \geqq n_0\}$ belongs to $\mathscr{F}$, we have $[(p_{k_n})] \in \mu(p)$. It follows from Theorem 2.1 that $\mu(p)$ is the unique monad containing $[(p_{k_n})]$.

Now let $(p_n)_{n=1}^{\infty}$ be not convergent to p. Then a number $\varepsilon>0$ and a subsequence $(p_{k_n})_{n=1}^{\infty}$ exists with $\varrho(p_{k_n}, p) \geqq \varepsilon$ $(n=1, 2, \ldots)$ Thus $[(p_{k_n})] \in m(p)$.

Similarly we can prove the following.

Theorem 3.2. Let $(p_n)_{n=1}^{\infty}$ be a sequence of points of (X, ϱ). If $[(p_n)] \in \mu(p)$, where $p \in X$, then there is a subsequence $(p_{k_n})_{n=1}^{\infty}$ of the sequence $(p_n)_{n=1}^{\infty}$ such that $(p_{k_n})_{n=1}^{\infty}$ converges to the point p.

Theorem 3.2. Let $(p_n)_{n=1}^{\infty}$ be a sequence of points of (X, ϱ). If $[(p_n)] \in \mu(p)$, a Cauchy sequence if and only if for any subsequence $(p_{k_n})_{n=1}^{\infty}$ we have $[(p_{k_n})] \approx [(p_n)]$.

Proof. Suppose $(p_n)_{n=1}^{\infty}$ to be a Cauchy sequence and $(p_{k_n})_{n=1}^{\infty}$ its subsequence. Then for any $\varepsilon>0$ there exists n_0 such that for $m, n \geqq n_0$ we have $\varrho(p_m, p_n)<\varepsilon$. If $n \geqq n_0$, then $k_n \geqq n_0$. Hence $\varrho(p_n, p_{k_n})<\varepsilon$. Since $\{n \mid n \geqq n_0\} \in \mathscr{F}$, we have $[(p_{k_n})] \approx [(p_n)]$.

Conversely, let $[(p_{k_n})] \approx [(p_n)]$ for any subsequence (p_{k_n}). Suppose $(p_n)_{n=1}^{\infty}$ not to be a Cauchy sequence. Then there exists a subsequence

$$p_{r_1}, p_{s_1}, p_{r_2}, p_{s_2}, \ldots, p_{r_n}, p_{s_n}, \ldots$$

such that $\varrho(p_{r_n}, p_{s_n}) \geqq \varepsilon$ $(n=1, 2, \ldots)$.

Since $[(p_{r_n})] \approx [(p_n)]$ there exists a set $F_1 \in \mathscr{F}$ such that $\varrho(p_{r_n}, p_n)<\frac{\varepsilon}{2}$ for $n \in F_1$.

Similarly $\varrho(p_{sn}, p_n)<\frac{\varepsilon}{2}$ on a set $F_2 \in \mathscr{F}$. The last two inequalities are satisfied on $F_1 \cap F_2 \in \mathscr{F}$. So for $n \in F_1 \cap F_2$

$$\varrho(p_{r_n}, p_{s_n}) \leqq \varrho(p_{r_n}, p_n)+\varrho(p_n, p_{s_n})<\varepsilon$$

It is a contradiction.

Remark 3.1. A nonstandard consideration how to see that a convergent sequence is a Cauchy sequence goes in the following manner. If $(p_n)_{n=1}^{\infty}$ converges to p, then for any subsequence (p_{k_n}) we have $[(p_{k_n})] \in \mu(p)$, $[(p_n)] \in \mu(p)$. So $[(p_{k_n})] \approx [(p_n)]$. Using Theorem 3.3 we obtain that $(p_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Theorem 3.4. A Cauchy sequence $(p_n)_{n=1}^{\infty}$ is convergent if and only if there exists a point $p \in X$ such that $[(p_n)_{n=1}^{\infty}] \in \mu(p)$.

Proof. The necessity is evident from Theorem 3.1. Let us prove the sufficiency. Take any subsequence $(p_{k_n})_{n=1}^{\infty}$. We have $[(p_{k_n})] \approx [(p_n)]$, because $(p_n)_{n=1}^{\infty}$ is a Cauchy sequence. Since $[(p_n)] \in \mu(p)$ we have $[p_{k_n}] \in \mu(p)$. The proof now follows by Theorem 3.1.

Corollary 3.1. A metric space (X, ϱ) is complete if and only if for any Cauchy sequence $(p_n)_{n=1}^{\infty}$ of points of X there exists a point $p \in X$ such that $[(p_n)] \in \mu(p)$.

Let us give a nonstandard formulation of the separability.

Theorem 3.5. A metric space (X, ϱ) is separable if and only if a countable set $A \subset X$ exists such that $\mu(p) \cap {}^*A \neq \emptyset$ for any $p \in X$.

Proof. Suppose X to be separable. Then there is a countable dense set $A \subset X$. So given $p \in X$ a sequence $[(p_n)_{n=1}^{\infty}]$ of points of A exists such that $(p_n)_{n=1}^{\infty}$ converges to p. So $[(p_n)] \in \mu(p)$. Evidently $[(p_n)] \in {}^*A$. Thus $\mu(p) \cap {}^*A \neq \emptyset$.

Now let $\mu(p) \cap {}^*A \neq 0$ for any $p \in X$. Take $\alpha \in \mu(p) \cap A$. We have $\alpha = [(p_n)]$ where $p_n \in A$ for any n belonging to a set $F_1 \in \mathscr{F}$.

Since $\alpha \in \mu(p)$ we obtain for any $\varepsilon > 0$ a set $F_2 \in \mathscr{F}$ such that $\varrho(p_n, p) < \varepsilon$ for $n \in F_2$. Thus for $n \in F_1 \cap F_2$ we have $p_n \in A$ and $\varrho(p_n, p) < \varepsilon$. Thus A is dense in X and the separability is proved.

Theorem 3.6. A subset $K \subset X$ where (X, ϱ) is a metric space is compact if and only if for any $p \in {}^*K$ there exists $q \in K$ such that $p \in \mu(q)$.

Proof. Let K be compact, $p \in {}^*K$. Susspose $p \notin \mu(q)$ for any $q \in K$. Then for any $q \in K$ there exists $\varepsilon_q > 0$ such that $p \notin {}^*O_{\varepsilon_q}(q)$. Evidently $K \subset \bigcup_{q \in K} O\varepsilon_q(q)$. By the compactness of K there exist $O_{\varepsilon_{qi}}(q_i)$, $i = 1, 2, \ldots, n$ such that $K \subset \bigcup_{i=1}^{n} O_{\varepsilon_{qi}}(q_i)$ which implies ${}^*K \subset \bigcup_{i=1}^{n} O^*_{\varepsilon_{qi}}(q_i)$. Consequently $p \notin {}^*K$. It is a contradiction.

Suppose now that for any $p \in {}^*K$ there is $q \in K$ with $p \in \mu(q)$. Let $(p_n)_{n=1}^{\infty}$ be a sequence of points of K. Take $p = [(p_n)]$. Then $p \in {}^*K$. So $p \in (q)$ for some $q \in K$. It follows from Theorem 3.2 that there exists a subsequence $(p_{k_n})_{n=1}^{\infty}$ converging to $q \in K$, proving that K is compact.

4. Some applications

All the theorems in Chapters 2 and 3 give a correspondence between a standard and nonstandard notions. They state always some necessary and sufficient condition. So they give a potential possibility to define the fundamental notions in the nonstandard metric spaces. It may be interesting if there is a good possibility to prove some other (of course well known) theorems in the nonstandard theory. Here we give some examples of such proofs.

Theorem 4.1. Any compact set $K \subset X$ is closed.

Proof. Suppose that for a point $p \in X\, \mu(p) \cap {}^*K \neq \emptyset$. Then there exists $\alpha \in \mu(p)$, $\alpha \in {}^*K$. Since K is compact there is $q \in K$ such that $\alpha \in \mu(q)$. So $\mu(p) \cap \mu(q) \neq \emptyset$. Thus (see Theorem 2.1) $p = q \in K$. The proof is finished.

Theorem 4.2. Let $(K_n)_{n=1}^{\infty}$ be a descending sequence of nonempty compact sets. Then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Proof. Let $p_n \in K_n$ for $n = 1, 2, \dots$ Take $(p_n)_{n=1}^{\infty}$. Evidently $\alpha = [(p_n)] \in {}^*K_n$ for $n = 1, 2, \dots$ Since K_n compact there exist $q_n \in K_n$ such that $\alpha \in \mu(q_n)$ for $n = 1, 2, \dots$ Thus $q_1 = q_2 = \dots$ Denoting $q = q_1 = \dots$, we have $q \in \bigcap_{n=1}^{\infty} K_n$. Thus $\bigcap_{n=1} K_n \neq \emptyset$.

Theorem 4.3. Let (X, ϱ) be a complete metric space. Let $(F_n)_{n=1}^{\infty}$ be a descending sequence of nonempty closed sets with diameters tending to zero. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Proof. Take $p_k \in F_k$ $(k = 1, 2, \dots)$. Then $\alpha = [(p_k)] \in {}^*F_n$ $(n = 1, 2, \dots)$. Since $(p_n)_{n=1}^{\infty}$ is a Cauchy sequence, there exists by Corollary 3.1 $p \in X$ such that $\alpha \in \mu(p)$. By Theoem 2.4 $p \in F_n$ $(n = 1, 2, \dots)$.

Theorem 4.4. Let $K \subset X$ be a compact set in a metric space (X, ϱ) and $F \subset K$ a closed set. Then F is compact.

Proof. Let $p \in {}^*F$. Then $p \in {}^*K$. Thus $p \in \mu(q)$ for some $q \in K$ and we have $\mu(q) \cap {}^*F \neq \emptyset$. Thus $q \in F$.

5. Extension of mappings. Hypermetric space

Let X_1, X_2 be nonempty sets and $X_1 \times X_2$ their product. If $A \subset X_1, \times X_2$, the extension was defined (definition 1.3). Thus ${}^*A = \{\alpha \mid \alpha = [(p_n, q_n)], (p_n, q_n) \in A$ for n belonging to a set $F \in \mathcal{F}\}$.

Taking $\alpha \in {}^*A$ we have a possibility to choose two elements $\alpha_1 = [(p_n)_{n=1}^{\infty}]$, $\alpha_2 = [(q_n)_{n=1}^{\infty}]$ such that $\alpha_1 \in {}^*X_1$, $\alpha_2 \in {}^*X_2$ and $(p_n, q_n) \in A$ for any n belonging to a set $F \in \mathcal{F}$ Evidently (α_1, α_2) is independent on the choise of $[(p_n, q_n)_{n=1}^{\infty}] \in \alpha$. The converse is also true i.e. if we tke $(\alpha_1, \alpha_2) \in {}^*X_1 \times {}^*X_2$ such that $\alpha_1 = [(p_n)]$, $\alpha_2 = [(q_n)]$ and $(p_n, q_n) \in A$ for n belonging to some $F \in \mathcal{F}$, we have $[(p_n, q_n)_{n=1}^{\infty}] \in {}^*A$. Thus we have a natural "isomorphism" between ${}^*A$ as defined in 1.3 and the set of all pairs (α_1, α_2) as constructed above. The situation is quite similar for the product $X_1 \times X_2 \times \dots \times X_n$. So there will be no confusion if we shall understand ${}^*A$ in the following slightly different form given in

Definition 5.1. Let $X_1, X_2, \dots, X_n$ be nonempty sets and ${}^*X_1, \dots, {}^*X_n$ their extensions (see definition 1.3). Let $A \subset X_1 \times X_2 \times \dots \times X_n$. Then we define

$$*A = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in *X_1 \times *X_2 \ldots *X_n \mid \alpha_1 = [(p_k^1)], \ldots,$$

$$\alpha_2 = [(p_k^2)], \ldots \alpha_n = [(p_k^n)], \text{ where } (p_k^1, p_k^2, \ldots, p_k^n) \in A$$

$$\text{for } k \text{ belonging to a set } F \in \mathcal{F}\}$$

Remark 5.1. Obviously we have $*(A_1 \times A_2 \times \ldots \times A_n) = *A_1 \times *A_2 \times \ldots *A_n$ if $A_i \subset X_i$ $(i = 1, 2, \ldots, n)$.

The set $X_1 \times X_2 \times \ldots \times X_n$ can be again naturaly embedded in its extension.

Theorem 5.1. Let $f \subset X_1 \times X_2$ be a mapping of X_1 into X_2. Then $*f$ is a mapping of $*X_1$ into $*X_2$. Moreover

$$*(D(f)) = D(*f), \quad *(H(f)) = H(*f),$$

where D and H denote the domain and range respectively.

Proof. Since f is a mapping of X_1 into X_2, we have $(a, b) \in f$, $(a, c) \in f$ implies $b = c$. Let (α, β), $(\alpha, \gamma) \in *f$ and $\alpha = [(a_n)]$, $\beta = [(b_n)]$, $\gamma = [(c_n)]$. Then in view of Definition 5.1 there exist $E, F \in \mathcal{F}$ such that for any $k \in E \cap F$ we have (a_k, b_k), $(a_k, c_k) \in f$. Hence $b_k = c_k$ for any $k \in E \cap F$. Since $E \cap F \in \mathcal{F}$, we have $\beta = \gamma$. Thus $*f$ is a mapping.

Further $\alpha \in D(*f)$ exactly if there is $\beta \in *X_2$ such that $(\alpha, \beta) \in *f$. This happens (according to Definition 5.1) if and only if for any sequences $(a_n)_{n=1}^{\infty} \in \alpha$, $(b_n)_{n=1}^{\infty} \in \beta$ there is $F \in \mathcal{F}$ such that $(a_k, b_k) \in f$ for any $k \in F$. Hence $a_k \in D(f)$ for any $k \in F$. The last happens if and only if $\alpha \in *(D(f))$. The proof $H(*f) = *(H(f))$ goes in a similar way.

Now let (X, ϱ) be a metric space. Since ϱ is a mapping of $X \times X$ into R we can prove in the same way as in Theorem 5.1 that $*\varrho$ is a mapping of $*(X \times X) = *X \times *X$ into $*R$ where $*R$ is the set of all hyperreal numbers. Suppose $p, q \in *X$, $\alpha \in *R$ such that $\alpha = *\varrho(p, q)$ i.e. $(p, q, \alpha) \in *\varrho$. By Definition 5.1 for any sequences $(p_n)_{n=1}^{\infty} \in p$, $(q_n)_{n=1}^{\infty} \in q$ $(a_n)_{n=1}^{\infty} \in \alpha$ there is $F \in \mathcal{F}$ such that for any $k \in F$ we have $(p_k, q_k, a_k) \in \varrho$. So $a_k = \varrho(p_k, q_k)$, proving that for any $p, q \in *X$ the number $*\varrho(p, q)$ is that one containing $(\varrho(p_n, q_n))_{n=1}^{\infty}$. From the last property one easily obtains the following:

(a^*) $*\varrho(p, q) \geqq 0$ for any $p, q \in *X$

(b^*) $*\varrho(p, q) = 0$ if and only if $p = q$

(c^*) $*\varrho(p, q) \leqq \varrho^*(p, r) + *\varrho(q, r)$ for any $p, q, r \in *X$.

Definition 5.2. Let (X, ϱ) be a metric space. Let $*X$, be extensions of X and $*\varrho$ respectively. The pair $(*X, *\varrho)$ is called a hypermetric space and it is said to be an extension of the metric space (X, ϱ). The function $*\varrho$ is called a hypermetric.

Theorem 5.2. Let (X, ϱ) be a metric space, $p \in X$ and $\varepsilon \in R^+ = (0, \infty)$. Let $O_\varepsilon(p) = \{q \in X \mid \varrho(p, q) < \varepsilon\}$. Then $*O(p) = \{q \in *X \mid *\varrho(p, q) < \varepsilon\}$.

Proof. The assertion $q \in *O_\varepsilon(p)$ is true if and only if for any sequence $(q_n)_{n=1}^{\infty} \in q$ there is $F \in \mathcal{F}$ such that $q_n \in O_\varepsilon(p)$ for any $n \in F$. The last holds if and only if $\varrho(q_n, p)<\varepsilon$ for any $n \in F$, which is equivalent to $*\varrho(p, q)<\varepsilon$.

Theorem 5.3. Let (X, ϱ) be a metric space and $(*X, *\varrho)$ its extension. Then for any $p \in X$

$\mu(p)=\{q \in *X \mid *\varrho(p, q)<\varepsilon$ for any real number $\varepsilon>0\}$.

Proof. According to Theorems 2.2 and 5.2 we have

$$\mu(p)=\bigcap_{\varepsilon>0} *O_\varepsilon(p)=\bigcap_{\varepsilon>0}\{q \in *X \mid *\varrho(p, q)<\varepsilon\}$$

6. Product of metric spaces

Given metric spaces (X_1, ϱ_1), (X_2, ϱ_2) their product is usually said to be the metric space $(X_1 \times X_2, \varrho)$ where ϱ is defined as

$$\varrho((p, q), (r, s))=\sqrt{\varrho_1^2(p, r)+\varrho_2^2(q, s)}$$

for any (p, q), $(r, s) \in X_1 \times X_2$.

There are of course other metrics on $X_1 \times X_2$ defining the same topology as the above metric i.e. the usual product topology (see e.g. [7]).

Take now the extension $(*X_1 \times *X_2, *\varrho)$ of the metric space $(X_1 \times X_2, \varrho)$. We are able to prove the following.

Theorem 6.1. Under the above assumptions

$$*\varrho((p, q), (r, s))=\sqrt{(*\varrho_1(p, r))^2+(*\varrho_2(q, s))^2}$$

for any

$$(p, q), (r, s) \in *X_1 \times *X_2.$$

Proof. *Take* $(p_n)_{n=1}^\infty \in p$, $(q_n)_{n=1}^\infty \in q$, $(r_n)_{n=1}^\infty \in r$, $(s_n)_{n=1}^\infty \in s$. Then $(*\varrho((p, q), (r, s))^2$ is the hyperreal number into which belongs the sequence $((\varrho((p_n, q_n), (r_n, s_n))^2)_{n=1}^\infty$ i.e. the sequence $(\varrho_1^2(p_n, r_n)+\varrho_2^2(q_n, s_n))_{n=1}^\infty$. By the definition of the sum and the product of hyperreal numbers we have

$$(\sqrt{\varrho_1^2(p_n, q_n)+\varrho_2^2(r_n, s_n)})_{n=1}^\infty \in \sqrt{(*\varrho_1(r, q))^2+(*\varrho_2(r, s))^2}$$

Since the square root of a hyperreal number $\alpha \geqq 0$ is the (unique) hyperreal number $\beta \geqq 0$ for which $\beta^2=\alpha$ we have

$$(\sqrt{\varrho_1^2(p_n, q_n)+\varrho_2^2(r_n, s_n)})_{n=1}^\infty \in \sqrt{(*\varrho_1(p, q))^2+(*\varrho_2(r, s))^2}$$

i.e.

$$^*\varrho((p, q), (r, s)) = \sqrt{(^*\varrho_1(p, q))^2 + (^*\varrho_2(r, s))^2}$$

It is now quite natural to denote $^*\varrho = {}^*\varrho_1 \times {}^*\varrho_2$.

Theorem 6.2. Let (X_1, ϱ_1), (X_2, ϱ_2) be metric spaces and $(X_1 \times X_2, \varrho_1 \times \varrho_2)$ their product. Let $(p, q) \in X_1 \times X_2$ $(r, s) \in {}^*(X_1 \times X_2)$. Then we have

$$(r, s) \in \mu((p, q)) \text{ if and only if } r \in \mu(p)$$

and simultaneously $q \in \mu(s)$.

Proof. By Theorem 5.3 $(r, s) \in \mu((p, q))$ exactly if $^*\varrho((p, q), (r, s)) < \varepsilon$ for any real $\varepsilon > 0$, the last being true if and only if $^*\varrho_1(p, r) < \varepsilon$ and simultaneously $^*\varrho_2(q, s) < \varepsilon$ for any $\varepsilon > 0$. The last two inequalities mean the same as $r \in \mu(p)$ and $s \in \mu(q)$.

Theorem 6.3. (Tichonoff's theorem). Let $(X_1 \varrho_1)$, (X_2, ϱ_2) be compact spaces. Then $(X_1 \times X_2, \varrho_1 \times \varrho_2)$ is compact.

Proof. Let $(r, s) \in {}^*X_1 \times {}^*X_2$. Then $r \in {}^*X_1$, $s \in {}^*X_2$. The last and the compactness of both X_1 and X_2 implies that $r \in \mu(p)$, $s \in \mu(q)$ where $p \in X_1$, $q \in X_2$. By Theorem 6.2 $((r, s)) \in \mu((p, q))$ and $(p, q) \in X_1 \times X_2$. So (see Theorem 2.5) $X_1 \times X_2, \varrho_1 \times \varrho_2)$ is a compact space.

7. Nonstandard formulation of the continuity

Let (X_1, ϱ_1), (X_2, ϱ_2) be metric spaces. The continuity in the nonstandard formulation may be motivated by the following

Theorem 7.1. A mapping f of the metric space (X_1, ϱ_1) into a metric space (X_2, ϱ_2) is continuous at a point $p \in X_1$ if and only if $^*f(\mu(p)) \subset \mu(f(p))$.

To prove the above Theorem we use the following.

Lemma 7.1. Let $f \subset X_1 \times X_2$ be a mapping of (X_1, ϱ_1) into (X_2, ϱ_2). Then for $A \subset D(f)$

$$^*(f(A)) = {}^*f(^*A)$$

The proof of Lemma 7.1 uses the same reasoning as that one of Theorem 5.1 and therefore will be omitted.

Proof of Theorem 7.1. Suppose f to be continuous at p. We have for any $\varepsilon > 0$ a positive δ such that $f(O_\delta(p)) \subset O_\varepsilon(f(p))$. Using this inclusion and Lemma 7.1 we obtain

$$^*(f(O_\delta(p)) = {}^*f(^*O_\delta(p)) \supset {}^*f\Big(\bigcap_{\delta>0} {}^*O_\delta(p)\Big) = {}^*f(\mu(p))$$

Thus

$$*f(\mu(p)) \subset *f(O_\delta(p)) \subset *O_\varepsilon(f(p))$$

for any $\varepsilon > 0$, hence

$$*f(\mu(p)) \subset \bigcap_{\varepsilon>0} *O_\varepsilon(f(p)) = \mu(f(p)).$$

Conversely, let $*f(\mu(p)) \subset \mu(f(p))$. Let $(p_n)_{n=1}^\infty$ be a sequence of points of X_1 converging to p. Then by Theorem 3.1 $[(p_{k_n})] \in \mu(p)$ for any subsequence $(p_{k_n})_{n=1}^\infty$

So $*f([(p_{k_n}])] \in \mu(f(p))$,

and we have

$$[f(p_{k_n})_{n=1}^\infty] \in *f([(p_{k_n})]) \in \mu(f(p))$$

Using again Theorem 3.1, we obtain that $(f(p_n))_{n=1}^\infty$ converges to $f(p)$.

As an application of the nonstandard formulation of the continuity let us give a nostandard proof of the following.

Theorem 7.2. The image $f(A)$ of a compact set under a continuous mapping f is a compact set.

Proof. Let $\beta \in *f((A)) = *f(*A)$. We have $\beta = *f(\alpha)$ where $\alpha \in *A$. Since A is compact $\alpha \in \mu(q)$ for some $q \in A$. Thus $\beta \in *f(\mu(q)) \subset \mu((f(q))$. The compactness of $(f(A))$ is proved.

Concluding remark. We mention e.g. [1] [3] [4] [6] as elementary approaches to nonstandard ideas of analysis.

REFERENCES

[1] Bednárik, J.—Riečanová, Z.: O nekonečne malých veličinách bez axióm. Matematické obzory, to appear.

[2] Davis, M.: Applied nonstandard analysis. New York, London, Sydney, Toronto 1977.

[3] Hatcher, W.: Calculus is algebra. Amer. Math. Monthly 1982, 362—370.

[4] Keisler, J. H.: Elementary calculus. Boston 1976.

[5] Kelley, J. L.: General Topology. New York 1955.

[6] Neubrunn, T.—Riečanová, Z.: Existujú nekonečne malé veličiny? Matematické obzory, 21/83, 7—17.

[7] Sikorski, R.: Funkcie rzecziwiste. Warszawa 1958.

[8] Stroyan, K. D.—Luxemburg, W.A.S.: Introduction to the theory of infinitesimals. New York 1976.

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SÚHRN

ELEMENTÁRNY NEŠTANDARDNÝ PRÍSTUP K METRICKÝM PRIESTOROM

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V článku sa ukazuje, ako možno budovať metódami neštandardnej analýzy teóriu metrických priestorov bez toho, aby bolo potrebné používať komplikované metódy matematickej logiky.

РЕЗЮМЕ

ЭЛЕМЕНТАРНЫЙ НЕСТАНДАРТНЫЙ ПОДХОД К МЕТРИЧЕСКИМ ПРОСТРАНСТВАМ

Т. Нойбрун—З. Риечанова, Братислава

Показывается, что теорию метрических пространств можно изложить методами нестандартного анализа не пользуясь сложными средствами математической логики.