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THE PROBLEM OF THE RAILWAY GOODS TRAFFIC

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I

We consider the problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{n(x)(t))}{x(t)}\right] = -\left(c_1 + \frac{c_2}{x(t)}\right) \tag{1}$$

$$x(0) = x_0 > 0, \tag{2}$$

where

$$n \tag{3}$$

is an increasing, continuous and positive function on $(0, \infty)$;

$$\frac{n(x)}{x} \tag{4}$$

is a decreasing and locally absolutely continuous function on $(0, \infty)$. This function maps the interval $(0, \infty)$ on (u_1, u_2), where

$$u_1 = \lim_{x\to\infty} \frac{n(x)}{x}$$

$$u_2 = \lim_{x\to 0^+} \frac{n(x)}{x}$$

and $0 < u_1 < u_2 (\leq \infty)$;

$$c_1 > 0, \quad c_2 > 0. \tag{5}$$

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The differential equation (1) can be interpreted in various transportation systems. For our purposes we shall use the terminology and the interpretation of the railway goods traffic.

$n(x)$ are the transporting charges of the transportation of the load weighing x ton for some unit distance. The function n is an increasing one, because the transportation of the heavier goods train is more expensive than that of a lighter one. On the other hand $\frac{n(x)}{x}$ must be a decreasing function else it would not have sense to accumulate the load to the heavy goods trains, but each ton of the comming load should be transported immediately.

In other words, the price of the transportation of one ton of the load in the heavier train is lower than in the lighter one (a number of engine drivers and train dispatchers does not change).

$c_1 > 0$ is a price of one hour of the idle time of one ton of the load.

$c_2 > 0$ is a price of one hour of the idle time of the train and railwaymen serving the train.

The solution $x = x(t)$ of the equation (1) gives such a dependency of the growth of the load weight on time. under which the charges of the idle time of the load and of the goods train are compensated with an effect following from the lower price of the transportation of one ton of the load in the heavier goods train.

The solution of (1) will be a curve serving for the decision how the load must grow in dependance on time, so that it would still be advantageous to wait for it.

II

The proof of the existence and of the uniqueness of the solution to the problem (1)—(2)

By the solution of the problem (1)—(2) we will understand (if there exists) any function x defined and continuous on $\langle 0, T)$ and such, that $\frac{n(x(t))}{x(t)}$ is locally absolutely continuous on $\langle 0, T)$ and x fulfils (2) and satisfies (1) almost everywhere in $\langle 0, T)$.

Let us denote $S(x) = \frac{n(x)}{x} \qquad (x \in (0, \infty))$.

Lemma 1. If there exists the solution of (1), it must be an increasing function.

Proof: According to our assumptions, the function S is the decreasing one. From (1) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[S(x(t))] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{n(x(t))}{x(t)}\right] =$$

$$= -\left(c_1 + \frac{c_2}{x(t)}\right) < 0 \qquad (t \in \langle 0, T)). \tag{6}$$

Thus the composite function $S(x(t))$ is a decreasing one. Hence x must be an increasing function, otherwise we would get $t_1, t_2 \in \langle 0, T)$ such that $t_1 < t_2$ and $x(t_1) \geqslant x(t_2)$. Then $S(x(t_1)) \leqslant S(x(t_2))$ which would be in controversy with (6).

In the next part we shall show that the problem (1)—(2) is equivalent to the problem

$$y'(t) = f(t, y(t)) \tag{7}$$

$$y(0) = 0, \tag{8}$$

where f will be given below.

The problem (1)—(2) can be transformed to the form

$$S(x(t)) = \frac{n(x(0))}{x(0)} - c_1 \cdot t - c_2 \cdot \int_0^t \frac{1}{x(s)} \, ds. \tag{9}$$

S is a decreasing and continuous function on $(0, \infty)$, thus there exists an inverse function S^{-1} defined on (u_1, u_2), this function being also a decreasing and continuous one.

Let us apply S^{-1} to (9). Then we get

$$x(t) = S^{-1}\left(\frac{n(x_0)}{x_0} - c_1 t - \int_0^t \frac{c_2}{x(s)} \, ds\right). \tag{10}$$

Let us assume that there exists a solution of the problem (1)—(2) and let us choose the substitution

$$y(t) = \int_0^t \frac{1}{x(s)} \, ds, \quad (0 \leqslant t < T). \tag{11}$$

Then $y'(t) = \frac{1}{x(t)}$ and in view of (10) we get

$$y'(t) = \frac{1}{S^{-1}(y_0 - c_1 t - c_2 y(t))} \tag{12}$$

where $y_0 = \frac{n(x_0)}{x_0}$. If we define the function

$$f(t, y) = \frac{1}{S^{-1}(y_0 - c_1 t - c_2 y)} \tag{13}$$

$(0 \leqslant t < \infty,\ u_1 < y_0 - c_1 t - c_2 y < u_2)$ then y fulfils (7), (8).

f is defined in the set $D = \{(t, y) \mid 0 \leqslant t < \infty,\ u_1 < y_0 - c_1 t - c_2 y < u_2\}$.

The following cases must be considered

(i) $u_2<\infty$. Then

$$\frac{y_0-u_2}{c_2}-\frac{c_1}{c_2}t<y<\frac{y_0-u_1}{c_2}-\frac{c_1}{c_2}t \tag{14}$$

$$(0\leq t<\infty).$$

(ii) $u_2=\infty$. Then

$$y<\frac{y_0-u_1}{c_2}-\frac{c_1}{c_2}t\ \ (0\leq t<\infty). \tag{15}$$

Lemma 2. The function x is a solution of the problem (1)—(2) on an interval $\langle 0, T)$ if and only if the function $y(t)=\int_0^t \frac{1}{x(s)}\,ds$ is a solution of the problem

$$y'(t)=f(t, y), \tag{7}$$

$$y(0)=0 \tag{8}$$

on the same interval and $f(t, y)$ is given by (13).

Proof: In the previous remark we have proved that if the function x is the solution of (1)—(2) defined on some interval, then the function y from (11) is the solution of the problem (7)—(8) defined on the same interval and f is given by (13).

On the other hand, let y be the solution of (7)—(8), where f is from (13). Then obviously the function

$$x(t)=\frac{1}{y'(t)}$$

(t from the interval of the definition of the function y) is the solution of the problem (1)—(2) on the same interval, on which y is defined.

Lemma 3. The point T of the interval $\langle 0, T)$ of the definition of the solution to the problem (1)—(2) fulfils

$$\frac{\dfrac{y_0-u_1}{c_2}}{\dfrac{1}{x_0}+\dfrac{c_1}{c_2}}<T\leq\frac{y_0-u_1}{c_1}. \tag{16}$$

Proof: For the interval of the definition of the solution of (1)—(2) by (9) we get

$$u_1<\frac{n(x(t))}{x(t)}=\frac{n(x_0)}{x_0}-c_1t-c_2\int_0^t\frac{1}{x(s)}\,ds<y_0-c_1t,$$

and so $t<\frac{y_0-u_1}{c_1}$ must be for $t\in\langle 0, T)$. Therefore, for the point T of the interval $\langle 0, T)$ of the definition of the solution to (1)—(2) holds

$$T\leqslant\frac{y_0-u_1}{c_1}. \tag{17}$$

In view of (14), (15) and (16) we may consider f in the triangle

$$D_f=\left\{(t, y)\left|0\leqslant t<\frac{y_0-u_1}{c_1},\ 0\leqslant y<\frac{y_0-u_1-c_1t}{c_2}\right.\right\}. \tag{18}$$

By Lemma 1 and (11) y is an increasing and strictly concave function, therefore $y(t)<\frac{t}{x_0}$ holds for $t\in(0, T)$ $\left(y'(t)=\frac{1}{x(t)},\ y'_+(0)=\frac{1}{x_0}\right.$ and a tangent to y in the poin $t=0$ has the form $\left.\frac{t}{x_0}\right)$.

By (18)

$$0\leqslant y(t)<\frac{y_0-u_1-c_1t}{c_2}\quad(0\leqslant t<T).$$

Thus the solution y gives us the following condition for T

$$\frac{T}{x_0}>\frac{y_0-u_1-c_1T}{c_2},$$

from which we get

$$T>\frac{\frac{y_0-u_1}{c_2}}{\frac{1}{x_0}+\frac{c_1}{c_2}}.$$

Lemma 4. The problem (7)—(8), where f is given from (13), has exactly one solution on the interval $\langle 0, T)$.

Proof: It is clear, that the function f is a continuous one, therefore there exists the solution of the problem (7)—(8) defined on $\langle 0, T)$. f is also a decreasing function of y, so by Peano's theorem there exists exactly one solution of (7)—(8).

Theorem 5. The problem (1)—(2) has exactly one solution x on $\langle 0, T)$, where T fulfils (16) and $\lim_{t\to T^-} x(t)=\infty$.

Proof: The first part of this theorem is obvious from the previous lemmas. We have to prove only $\lim_{t\to T^-} x(t)=\infty$.

In view of (11), (7) and (13) we get

$$\lim_{t\to T^-} x(t)=\lim_{t\to T^-}\frac{1}{y'(t)}=\lim_{t\to T^-} S^{-1}(y_0-c_1t-c_2y(t))\,.$$

The function y is an increasing, concave and locally absolutely continuous one, therefore in view of (18), (14) and (15) we get

$$\lim_{t\to T^-} y(t)=\left[\frac{y_0-u_1}{c_2}-\frac{c_1t}{c_2}\right]_{t=T}=\frac{y_0-u_1-c_1T}{c_2}$$

and on (13) we get

$$u_1=y_0-c_1T-c_2\lim_{t\to T^-} y(t)\,.$$

Thus

$$\lim_{u\to u_1^+} S^{-1}(u)=S^{-1}(y_0-c_1T-c_2\lim_{t\to T^-} y(t))=\infty$$

and

$$\lim_{t\to T^-} S^{-1}(y_0-c_1t-c_2y(t))=S^{-1}(y_0-c_1T-c_2\lim_{t\to T^-} y(t))=\infty\,,$$

from which it is evident, that $\lim_{t\to T^-} x(t)=\infty$.

III

Approximate solution of the problem (1)—(2)

In the previous part we have proved that there exists exactly one solution of the problem (1)—(2) on the interval $\langle 0, T)$, where T is determined by (16).

In the next part we shall look for an approximate solution of the problem (1)—(2).

On Theorem 5 the problem (1)—(2) is equivalent to the problem (7)—(8), where the relation between the solutions of the problem (7)—(8) and the problem (1)—(2) is determined by (13).

Let us choose $\eta>0$ and let us consider the problem (7)—(8) in $\langle 0, T-\eta\rangle$. The function $f=f(t, y)$ is a continuous one on the rectangle

$$V=\langle 0, T-\eta\rangle\times\left\langle 0, \frac{T-\eta}{x_0}\right\rangle,$$

because

$$y(t)\leqslant\int_0^{T-\eta}\frac{1}{x(s)}\,ds\leqslant\frac{T-\eta}{x_0}\quad(t\in\langle 0, T-\eta\rangle).$$

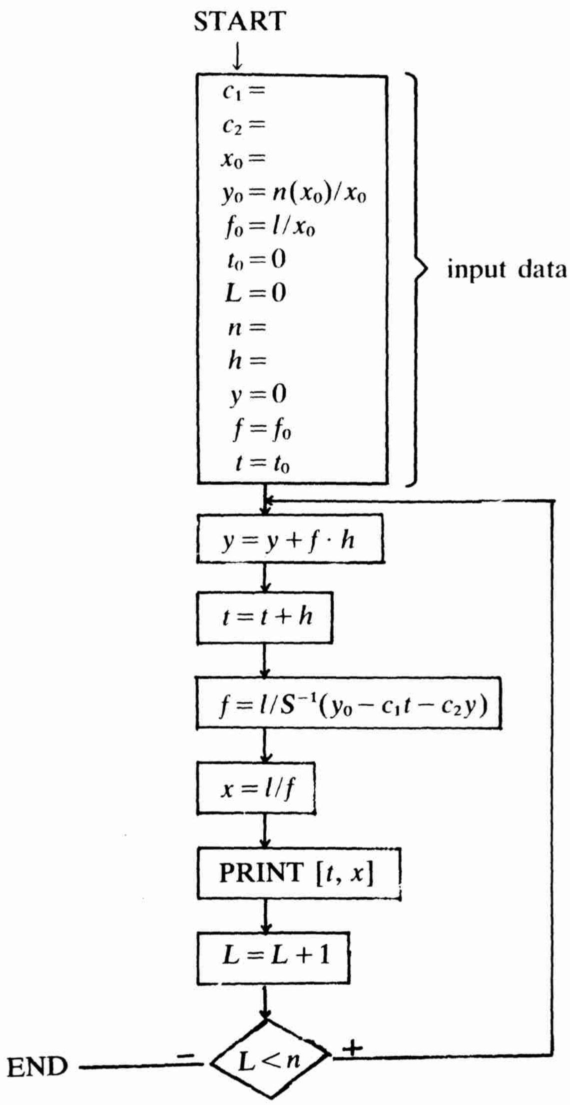
START
$c_1 =$
$c_2 =$
$x_0 =$
$y_0 = n(x_0)/x_0$
$f_0 = l/x_0$
$t_0 = 0$
$L = 0$
$n =$
$h =$
$y = 0$
$f = f_0$
$t = t_0$
input data
$y = y + f \cdot h$
$t = t + h$
$f = l/S^{-1}(y_0 - c_1 t - c_2 y)$
$x = l/f$
PRINT $[t, x]$
$L = L + 1$
$L < n$
$-$
$+$
END

Figure 1

Therefore the function f is a bounded one on V with some constant. Let us sign it K.

Let $c = \min\left\{T - \eta, \frac{T-\eta}{x_0 K}\right\}$. Then there exists the Euler's polygon, which is an ε-approximate solution of the problem (7)—(8). We shall get the approximate solution of the problem (1)—(2) on the interval $\langle 0, c\rangle$ from the relation (11).

In the next part we shall give a flow sheet for the solution of this problem on a computer.

Let us choose a step h, so that a division of the interval $\langle 0, c\rangle$ given by dividing points

$$t_0 = 0 < t_1 = h < t_2 = 2h < \ldots < t_n = c$$

would be equidistant and sufficiently fine. From the construction of the Euler's polygon

$$y(t) = y(t_k) + f(t_k, y(t_k)) \cdot (t - t_k) \ (t \in \langle t_k, t_{k+1}\rangle)$$

we get the value of y in the point t_{k+1}

$$y(t_{k+1}) = y(t_k) + f(t_k, y(t_k)) \cdot h .$$

Let us denote $y(t_k) = y_k$, $f(t_k, y_k) = f_k$. Then we get

$$y_{k+1} = y_k + f_k h \quad \text{and} \quad x(t_k) = x_k = \frac{1}{f_k}.$$

The flow sheet is in *Fig. 1.* It is clear, that if the function n is given from practice, then the programmer will be able solve this problem on a computer.

As a result from the computer we shall get a table of the values of the function x. Then, if in practice the train dispatcher has this table, he will easily decide whether the goods train can leave a station, or it is more advantageous to wait for additional load.

We have to note that in solving of this problem for the concrete function n given on the interval $\langle 0, T)$, the real time interval must be mapped on to on $\langle 0, T)$. The flow sheet in Fig. 1. must then be regulated in view of this note.

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SÚHRN

PROBLÉM Z NÁKLADNEJ ŽELEZNIČNEJ DOPRAVY

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V práci sa dokazuje existencia a jednoznačnosť riešenia pre počiatočnú úlohu, ktorá má interpretáciu v doprave.

РЕЗЮМЕ

ПРОБЛЕМА ИЗ ЖЕЛЕЗНОДОРОЖНОГО ТРАНСПОРТА

М. Франьова, Братислава

В работе доказывается существование и единственность решения задачи с интерпретацией в железнодорожном транспорте товарных поэздов.

DIDAKTIKA MATEMATIKY