UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
XLIV—XLV—1984

GALERKIN'S APPROXIMATION AND EXISTENCE THEOREMS FOR A NONLINEAR BOUNDARY VALUE PROBLEM IN ORDINARY DIFFERENTIAL EQUATIONS

ZDENEK SCHNEIDER, Bratislava

Consider the nonlinear boundary value problem

$$x''(t)+f(t, x(t))=0$$
$$A_0x(0)+A_1x'(0)=0$$
$$B_0x(l)+B_1x'(l)=0.$$

(A_0, A_1, B_0, B_1 are real constants.)

The aim of this paper is to formulate sufficient conditions for

1. the existence of a solution $y(t)$ for this problem
2. the existence of Galerkin's approximation $x_G(t)$ for $y(t)$
3. an error bound of the difference $y(t)-x_0(t)$ for given $x_0(t)$.

1. Backkground

It is well known [1] that the linear boundary value problem ($l>0$)

$$x''(t)+\lambda x(t)=0 \tag{1.1}$$

$$A_0x(0)+A_1x'(0)=0 \qquad |A_0|+|A_1|>0 \tag{1.2}$$

$$B_0x(l)+B_1x'(l)=0 \qquad |B_0|+|B_1|>0 \tag{1.3}$$

has exactly countable many real eigenvalues $\lambda_1, \ldots, \lambda_j, \ldots$. For λ_j there exists a corresponding nontrivial solution of (1.1) (1.2) (1.3) — the eigenfunction $\varphi_j(t)$ which is unique up to a constant multiple. It holds

1. $\lim\limits_{j\to\infty} \lambda_j = +\infty$

2. the system $\{\varphi_i(t)\}$ is a complete orthogonal system in the real separable Hilbert space $S \equiv L^2(0, l)$.

The scalar product $(.,.)$ and the norm $\|.\|$ in S are defined as usually. Without loss of generality we can suppose that the system $\{\varphi_i(t)\}$ is an orthonormal system in $L^2(0, l)$.

For each $x(t) \in L^2(0, l)$ let $\sum_{j=1}^{\infty} x_j\varphi_j(t)$ be its Fourier series with $x_j = (x(t), \varphi_j(t))$.

For a bounded function $x(t) \in S$ let

$$\mu(x) = \sup_{\langle 0, l\rangle} |x(t)| \tag{1.4}$$

If $\lambda = 0$ is not an eigenvalue of (1.1) (1.2) (1.3), then the series

$$\sum_{j=1}^{\infty} \lambda_j^{-1}, \quad \sum_{j=1}^{\infty} \lambda_j^{-2},$$

are convergent [6].

The number $\lambda = 0$ is an eigenvalue of the problem (1.1) (1.2) (1.3) if and only if the boundary conditions are of the form

$$\begin{aligned}
&\text{(i)} \quad x'(0) = 0 \\
&\qquad\ \ x'(l) = 0
\end{aligned}$$

$$\begin{aligned}
&\text{(ii)} \quad A_0x(0) + A_1x'(0) = 0 \qquad A_0 \cdot A_1 \neq 0 \\
&\qquad\ \ x(l) = 0 \\
&\qquad\ \ \text{and } A_0l - A_1 = 0,
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
&\text{(iii)} \quad x(0) = 0 \\
&\qquad\ \ B_0x(l) + B_1x'(l) = 0 \qquad B_0 \cdot B_1 \neq 0 \\
&\qquad\ \ \text{and } B_0l + B_1 = 0,
\end{aligned} \tag{1.6}$$

$$\begin{aligned}
&\text{(iv)} \quad A_0x(0) + A_1x'(0) = 0 \qquad A_0 \cdot A_1 \neq 0 \\
&\qquad\ \ B_0x(l) + B_1x'(l) = 0 \qquad B_0 \cdot B_1 \neq 0 \\
&\qquad\ \ \text{and } A_0(B_0l + B_1) - A_1B_0 = 0.
\end{aligned} \tag{1.7}$$

In the case (i) the positive λ_j are

$$\lambda_j = (j\pi/l)^2 \quad j = 1, 2, \dots. \tag{1.8}$$

In the cases (ii) and (iii) the positive λ_j are positive roots of the equation

$$\xi = l\sqrt{\lambda}, \text{ where } \operatorname{tg}(\xi) = \xi, \tag{1.9}$$

and it holds $j\pi < \xi_j < j\pi + \pi/2$, $j = 1, 2, \dots$.

Evidently, the series $(\lambda_j > 0)$

$$\sum_{j=1}^{\infty} \lambda_j^{-1}, \quad \sum_{j=1}^{\infty} \lambda_j^{-2}$$

for λ_j determined by the relations (1.8) (1.9) are convergent. In the case (iv) using Theorem 8 ([1] p. 270) on the asymptotic distribution of the positive eigenvalues of the regular Sturm—Liouville system, it can be proved that the series $(\lambda_j > 0) \sum_{j=1}^{\infty} \lambda_j^{-1}$, $\sum_{j=1}^{\infty} \lambda_j^{-2}$ are convergent.

For $\lambda_j > 0$ the corresponding eigenfunction $\varphi_j(t)$ is of the form

$$a_j \cos(\sqrt{\lambda_j} t) + b_j \sin(\sqrt{\lambda_j} t) \tag{1.10}$$

Lemma 1.1.

There exists a positive number A such that for the coefficients a_j, b_j from (1.10) the relation

$$|a_j| + |b_j| \leqq A, \quad j = 1, 2, \ldots \tag{1.11}$$

holds.

We can prove Lemma 1.1 using the orthonormality of $\{\varphi_j(t)\}$.

2. Basic conception

Let us consider a nonlinear boundary value problem

$$x''(t) + f(t, x(t)) = 0 \tag{2.1}$$

$$A_0 x(0) + A_1 x'(0) = 0 \qquad |A_0| + |A_1| > 0 \tag{2.2}$$

$$B_0 x(l) + B_1 x'(l) = 0 \qquad |B_0| + |B_1| > 0 \tag{2.3}$$

in a real separable Hilbert space $S = L^2(0, l)$.

The linear boundary value problem (1.1) (1.2) (1.3) with the same boundary conditions as in (2.1) (2.2) (2.3) is usually called the associate linear boundary value problem to the problem (2.1) (2.2) (2.3).

Many sufficient conditions given by several authors are known for the existence of a solution of (2.1) (2.2) (2.3). Some of them are given in Cesari's papers [2] [3].

Let us consider an operator equation $Kx \equiv Ex - Nx = 0$ with E linear operator and N nonlinear operator. The main idea about solving this operator equation is proposed by Cesari [2]. Essentially this method consists in splitting of $Kx = 0$ into an equivalent system $x = Tx$, $P(Kx) = 0$ of two equations. The equation $x = Tx$ is called auxiliary equation, the other equation is called bifurcation or determining equation.

Let S be a direct sum of the subspaces S_0, S_1 i. e. $S = S_0 \oplus S_1$, and let P, H be the operators with the following properties:

P is a projection, $P\colon S \to S_0$, $\mathcal{R}(P) = S_0$, $\mathcal{N}(P) = S_1$, $Px \in \mathcal{D}(E)$, $H\colon S_1 \to S_1$, $H(I-P)Nx \in \mathcal{D}(E)$ for $x \in \mathcal{D}(N)$ and

$$\begin{aligned} &H(I-P)Ex = (I-P)x && x \in \mathcal{D}(E) \\ &PEx = EPx && x \in \mathcal{D}(E) \\ &EH(I-P)Nx = (I-P)Nx && x \in \mathcal{D}(N). \end{aligned} \tag{2.4}$$

Then the problem $Ex = Nx$ is equivalent to the system of two equations

$$\begin{gathered} x = Px + H(I-P)Nx \equiv Tx \\ P(Ex - Nx) = 0. \end{gathered}$$

In our case let us define the operators E, N in the following way:

$$E\colon \mathcal{D}(E) \subset S \to S \quad E(x)(t) = x''(t) \quad t \in (0, l)$$

$$\begin{gathered} \mathcal{D}(E) = \{x \in S \mid A_0 x(0) + A_1 x'(0) = 0,\ B_0 x(l) + B_1 x'(l) = 0,\ x(t) \in C^{(1)}(\langle 0, l\rangle), \\ x''(t) \in L^2(0, l)\} \end{gathered}$$

and

$$\begin{gathered} N\colon \mathcal{D}(N) \subset S \to S,\ N(x)(t) = -f(t, x(t))\ t \in \langle 0, l\rangle, \\ \mathcal{D}(N) = C(\langle 0, l\rangle). \end{gathered}$$

Let

(p1) $f(t, \xi)$ be continuous and Lipschitzian, in variable ξ and with constant L, function on $\langle 0, l\rangle \times \mathbf{R}$.

From (p1) immediately it follows that

$$\|Nx - Nz\| \leqq L\|x - z\| \text{ for } x, z \in \mathcal{D}(N).$$

Let

(p2) m be a natural number such that

a) for the associate linear boundary value problem (1.1) (1.2) (1.3) is $\lambda_j > 0$ for $j > m$

b) $L \cdot \lambda_{m+1}^{-1} < 1$ (2.5)

For this number m let

$$S_0 = \text{span}\,(\varphi_1(t), \ldots, \varphi_m(t)),$$

$$S_1 = \text{span}(\varphi_{m+1}(t), \ldots)$$

be closed subspaces. (The eigenfunctions $\varphi_j(t)$ are determined by the associate linear boundary value problem (1.1) (1.2) (1.3).) Obviously, $S_0 \oplus S_1 = S$ and $S_0 \subset \mathcal{D}(E)$.

For $x \in S$ let $\sum_{j=1}^{\infty} x_j \varphi_j(t)$ be the associated Fourier series.

Let the operators P, $I-P$, H be defined as follows:

$$P: S \to S_0, \quad P(x)(t) = \sum_{j=1}^{m} x_j \varphi_j(t),$$

$$I-P: S \to S_1, \quad (I-P)(x)(t) = \sum_{j=m+1}^{\infty} x_j \varphi_j(t).$$

Operators P, $I-P$ are projections of S.

$$H: S_1 \to S_1, \quad H(x)(t) = -\sum_{j=m+1}^{\infty} \lambda_j^{-1} x_j \varphi_j(t).$$

Lemma 2.1.

H is a linear operator and $H: S_1 \to \mathcal{D}(E)$.
Sketch of the proof.

Let $g(t) \equiv H(x)(t) = -\sum_{j=m+1}^{\infty} \lambda_j^{-1} x_j (a_j \cos(\sqrt{\lambda_j} t) + b_j \sin(\sqrt{\lambda_j} t))$.

From Weierstrass's theorem and Cauchy's—Schwarz's inequality it follows that $g(t)$, $g'(t)$ are continuous functions on $\langle 0, l \rangle$. It holds

$$g'(t) = -\sum_{j=m+1}^{\infty} \lambda_j^{-1/2} x_j (-a_j \sin(\sqrt{\lambda_j} t) + b_j \cos(\sqrt{\lambda_j} t)).$$

The eigenfunctions $\varphi_j(t)$ satisfy the boundary conditions (2.2) (2.3), thus $g(t)$ also satisfies the boundary conditions (2.2) (2.3). Let us define $v(t)$ as follows

$$v(t) = -\sum_{j=m+1}^{\infty} x_j (-a_j \cos(\sqrt{\lambda_j} t) - b_j \sin(\sqrt{\lambda_j} t)).$$

Obviously, $v(t) \in L^2(0, l) \subset L(0, l)$ and so we can define the function

$$h(t) = g'(0) + \int_0^t v(s)\, ds \quad t \in \langle 0, l \rangle.$$

Using Theorem IX.3.4. [7] p. 247 it is easy to see that $g'(t) = h(t)$ for $t \in \langle 0, l \rangle$, thus $g''(t)$ exists a.e. and belongs to $L^2(0, l)$. Q.E.D.

It is apparent that the relations

$$\begin{array}{ll} H(I-P)Ex = (I-P)x & x \in \mathcal{D}(E) \\ EPx = PEx & x \in \mathcal{D}(E) \\ EH(I-P)Nx = (I-P)Nx & x \in \mathcal{D}(N) \end{array}$$

are valid and so the problem $Ex = Nx$ is equivalent to the system of two equations

$$x = Px + H(I-P)Nx$$

$$P(Ex - Nx) = 0.$$

3. Auxiliary equation

Let $x \in S$, then using Cauchy's—Schwarz's inequality and standard technique of mathematical analysis, it is easy to see that the relations

$$\|H(I-P)x\| \leqq k \cdot \|x\|$$

$$\mu(H(I-P)x) \leqq k' \cdot \|x\|$$

are valid for

$$k = \lambda_{m+1}^{-1} \tag{3.1}$$

$$k' = A \cdot \left(\sum_{j=m+1}^{\infty} \lambda_j^{-2} \right)^{1/2} \tag{3.2}$$

with the constant A determined in Lemma 1.1.

Let

$$x_0(t) = \gamma_{01}\varphi_1(t) + \ldots + \gamma_{0m}\varphi_m(t) \tag{3.3}$$

be a fixed element from S_0. (The initial — given or computed approximation for the "solution" $y(t)$.)

Let us denote $\Delta = H(I-P)(Ex_0 - Nx_0)$.

If

$$f(t, x_0(t)) = w(t) \tag{3.4}$$

and

$$w(t) = \sum_{j=1}^{\infty} w_j\varphi_j(t), \text{ then } \Delta = -\sum_{j=m+1}^{\infty} \lambda_j^{-1} w_j \varphi_j(t),$$

and so the bounds

$$\|\Delta\| \leqslant \lambda_{m+1}^{-1} \cdot \|w\|, \quad \mu(\Delta) \leqslant A \cdot \left(\sum_{j=m+1}^{\infty} \lambda_j^{-2} \right)^{1/2} \cdot \|w\|$$

are valid.

Let us denote

$$b = \lambda_{m+1}^{-1} \|w\| \tag{3.5}$$

$$b' = A \cdot \left(\sum_{j=m+1}^{\infty} \lambda_j^{-2} \right)^{1/2} \cdot \|w\| \tag{3.6}$$

Using Theorem iv [2] p. 393: if there exist numbers c, d, r, R_0 so that

$$\begin{aligned} &\text{(i) } O<c<d, \ O<r<R_0 \\ &\text{(ii) } kL<1 \\ &\text{(iii) } kLd<d-c-b \\ &\text{(iv) } k'Ld<R_0-r-b' \end{aligned} \tag{3.7}$$

(v) the implication $z \in S_0$, $\|z\| \leq c \Rightarrow \mu(z) \leq r$ holds, then the operator $Tx = Px + H(I-P)Nx$ is contractive mapping on complete set $S_0^* = \{x \in S \mid Px = Px^*, \|x - x_0\| \leq d, \mu(x - x_0) \leqq R_0\}$, where x^* is an arbitrary, but fixed element from the set $V = \{x \in S_0 \mid \|x - x_0\| \leqq c\}$.

Remember that the choice of the number m in (2.5) is such that the inequality (ii) in (3.7) is satisfied.

Let us denote

$$B = \max_{1 \leqq j \leqq m} \left(\sup_{\langle 0, l \rangle} |\varphi_j(t)| \right) \tag{3.8}$$

By easy calculations we can prove

Lemma 3.1.

If for a $c>0$ we have

$$d > (c+b)/(1-kL) \tag{3.9}$$

$$r = B \cdot \sqrt{m} \cdot c \tag{3.10}$$

$$R_0 > k'Ld + r + b', \tag{3.11}$$

then the conditions (i), (iii), (iv), (v) from (3.7) are satisfied.

Thus the equation $x = Px + H(I-P)Nx$ has exactly one solution $x(x^*) \in S_0^*$ if the conditions (2.5), (3.9), (3.10), (3.11) are satisfied. This solution depends continuously on $x^* \in V$. From Lemma 2.1 we see that $x(x^*) \in \mathcal{D}(E) \cap \mathcal{D}(N)$ and so $x(x^*)$ will be the solution of the problem $Kx = 0$ if and only if

$$P(E(x(x^*)) - N(x(x^*))) = 0 \tag{3.12}$$

4. Determining equation

The system of equations (3.12) is in [2] analyzed by considerations based on the degree of a mapping [4].

Let us denote, for given m determined by (2.5) and $w(t)$ by (3.4),

$$x(t)=\gamma_1\varphi_1(t)+\ldots+\gamma_m\varphi_m(t), \quad \gamma=(\gamma_1, \ldots, \gamma_m)$$

$$u_i(\gamma)\equiv(Kx, \varphi_i)=-\lambda_i\gamma_i+\int_0^l f(t, x(t))\varphi_i(t)\,\mathrm{d}t \tag{4.1}$$

$$\eta_i\equiv(Kx_0, \varphi_i)=-\lambda_i\gamma_{0i}+w_i, \quad i=1, \ldots, m \tag{4.2}$$

$$\eta=(\eta_1^2+\ldots+\eta_m^2)^{1/2} \tag{4.3}$$

With this notation the Galerkin's approximation $x_G(t)=\gamma_{G1}\varphi_1(t)+\ldots+\gamma_{Gm}\varphi_m(t)$ of the solution $y(t)$ is determined by the nonlinear system of equations

$$u_i(\gamma_G)=0, \quad i=1, \ldots, m \tag{4.1.G}$$

We can distinguish two cases $m=1$ and $m>1$. For $m=1$ we have only one nonlinear equation in (3.12) and for $m>1$ we have a system of m nonlinear equations.

The case $m=1$.

Let

(p3) there exist a number $c^*>0$ such that for numbers
$g_1=(K((\gamma_{01}+c^*)\varphi_1), \varphi_1)\equiv u_1(\gamma_{01}+c^*)$
$g_2=(K((\gamma_{01}-c^*)\varphi_1), \varphi_1)\equiv u_1(\gamma_{01}-c^*)$
d determined by (3.9) for $c=c^*$
b determined by (3.5)
it holds
$\operatorname{sgn}(g_1\cdot g_2)<0$
$L(\lambda_2^{-1}Ld+b)<\min(|g_1|, |g_2|)$.

Then following Theorem viii [2] p. 398 it holds

Theorem 4.1. $(m=1)$

Suppose that hypotheses (p1), (p2) (for $m=1$), (p3) are satisfied. Then there
1. exists a solution $y(t)$ of the equation $Ex-Nx=0$,
2. exists a Galerkin's approximation $x_G(t)$ for $y(t)$ and it holds

$$\|y-x_0\|\leqq d, \ \|Py-x_0\|\leqq c, \ \mu(y-x_0)\leqq R_0, \ \mu(Py-x_0)\leqq r.$$

Remark.

The element $x_0(t)$ is given by (3.3), the numbers r, R_0 are determined by (3.10) and (3.11).

In the paper [5] it was shown how this theorem can be used to establish existence theorem for certain Fučik's nonlinear boundary value problem.

The case $m>1$.

Let

(p4) functions $f_\xi(t, \xi), f_{\xi\xi}(t, \xi)$ exist and be continuous on $\langle 0, l\rangle\times\mathbf{R}$.

Let

(p5) for given $x_0(t)$, $\mathcal{A}=(a_{ij})$ be a regular matrix with elements

$$a_{ij}=\left(\frac{\partial u_i(\gamma)}{\partial \gamma_j}\right)\Bigg|_{\gamma=(\gamma_{01},\cdots,\gamma_{0m})}=$$

$$=\delta_{ij}(-\lambda_i)+\int_0^l f_x(t,\,x_0(t))\varphi_i(t)\varphi_j(t)\,\mathrm{d}t \quad i,j=1,\,\ldots,\,m \tag{4.4}$$

If $\mathcal{B}=\mathcal{A}^{-1}$, let $\mathcal{C}=(\mathcal{B}^{\mathrm{T}}\cdot\mathcal{B})$. If $\sigma_1,\,\ldots,\,\sigma_m$ are the (necessary positive) roots of the equation $\det(\mathcal{C}-\sigma \mathrm{I})=0$, let

$$\sigma=\min(\sigma_1^{-1/2},\,\ldots,\,\sigma_m^{-1/2}). \tag{4.5}$$

Furthermore denote

$$M(c)=\max_{\gamma\in\bar{G}}\left|\frac{\partial^2 u_i(\gamma)}{\partial\gamma_j\,\partial\gamma_n}\right| \quad i,\,j,\,n=1,\,\ldots,\,m, \tag{4.6}$$

where $\bar{G}=\sum_{j=1}^{m}(\gamma_j-\gamma_{0j})^2\leqslant c^2$ for some, yet undetermined, $c\in\mathbf{R}$, $c>0$.

Firstly let us consider the case $M(c)=0$ for arbitrary $c>0$.

Let

(p6) for given $x_0(t)$ it hold $\sigma>kL^2/(1-kL)$

(p7) $M(c)=0$ for arbitrary $c>0$.

Theorem 4.2. $(M(c)=0)$

Suppose that for given $x_0(t)$ hypotheses (p1), (p2) (with $m>1$), (p4), (p5), (p6), (p7) are satisfied. Then there

1. exists a solution $y(t)$ of the equation $Ex-Nx=0$,
2. exists a Galerkin's approximation $x_G(t)$ for $y(t)$ and it holds

$$\|y-x_0\|\leqslant d,\ \|Py-x_0\|\leqslant c^*,\ \mu(y-x_0)\leqq R_0,\ \mu(Py-x_0)\leqslant r$$

$$\text{for}\qquad c^*>\frac{(1-kL)\eta+Lb}{(1-kL)\sigma-kL^2}.$$

Proof.

From Theorem viii [2] p. 398 it follows that it is sufficient to show that there exists a number c^* such that $L(kLd+b)<c^*\sigma-\eta$ for d determined by (3.9). By (3.9) we see that the inequality

$$\frac{c^*+b}{1-kL}<\frac{c^*\sigma-\eta-Lb}{kL^2} \tag{4.7}$$

must be satisfied. Using (p6) it clearly follows that the condition (4.7) is satisfied if

$$c^*>\frac{(1-kL)\eta+Lb}{(1-kL)\sigma-kL^2}$$

and the proof of the theorem is concluded. (Q.E.D.

Secondly let $M(c)$, for given $x_0(t)$, as a function c, be bounded on $(0, \infty)$. More precisely

$$\text{(p8)}\quad M(c) < \frac{(\sigma(1-kL)-kL^2)^2}{2\cdot m^{(3/2)}\cdot(1-kL)\cdot(\eta\cdot(1-kL)+Lb)}$$

Theorem 4.3.

Suppose that for given $x_0(t)$ hypotheses (p1), (p2) (with $m>1$), (p4), (p5), (p6), (p8) are satisfied. Then there

1. exists a solution $y(t)$ of the equation $Ex-Nx=0$
2. exists a Galerkin's approximation $x_G(t)$ for $y(t)$ and it holds

$$\|y-x_0\|\leq d,\ \|Py-x_0\|\leq c^*,\ \mu(y-x_0)\leqq R_0,\ \mu(Py-x_0)\leq r$$

for some $c^*>0$.

Proof.

From Theorem viii [2] p. 398 it follows that it is sufficient to show that there exists number $c^*>0$ such that

$$L(kLd+b)<c^*\sigma-2^{-1}\cdot M(c^*)\cdot m^{(3/2)}\cdot(c^*)^2-\eta$$

for d determined by (3.9) with $c=c^*$. From (3.9) we see that the inequality

$$\frac{c^*+b}{1-kL}<\frac{c^*-2^{-1}\cdot M(c^*)\cdot m^{(3/2)}\cdot(c^*)^2-\eta-Lb}{kL^2} \tag{4.8}$$

must be satisfied. Using (p6) and (p8) clearly follows that for some $c^*>0$ the inequality is satisfied. Q.E.D.

In previous theorems it is stated that there exists Galerkin's approximation $x_G(t)$ for the solution $y(t)$. That means that the nonlinear system of equations (4.1G) for coefficients $\gamma_{G1}, \ldots, \gamma_{Gm}$ has (at least one) solution. In a concrete case we can also determine bounds for $\|y(t)-x_G(t)\|$ and $\mu(y(t)-x_G(t))$. We can use previous theorems with $x_0(t)=x_G(t)$. Of course for the first approximation of the „solution" we did not have to choose $x_0(t)$ but some Galerkin's approximation if we were able to solve the nonlinear system of equations (4.1.G).

5. An example

Let us consider the problem

$$x''(t)+L\sin(x(t))+p(t)=0 \tag{5.1}$$

$$x'(0)=x'(1)=0. \tag{5.2}$$

where $L>0$ and $p(t)\in C(\langle 0,1\rangle)$.

For the associate linear boundary value problem it is

$$\lambda_1=0,\ \lambda_j=((j-1)\pi)^2 \quad j=2, 3, \ldots$$

$$\varphi_1(t)=1,\ \varphi_j(t)=\sqrt{2}\cos((j-1)\pi t) \quad j=2, 3, \ldots.$$

For A in Lemma 1.1 we have $A=\sqrt{2}$.

The nonlinear part of the problem (5.1) (5.2) is Lipschitzian with the constant $L>0$.

We choose $m=m_0\geqq 2$ arbitrary, but fixed. In (2.5) it is

$$k=\lambda_{m_0+1}^{-1}=1/(m_0\pi)^2 \tag{5.3}$$

thus let us consider the problem (5.1)(5.2) for $L<(m_0\pi)^2$.

For k' in (3.2) we get

$$k'=\sqrt{2}\cdot(1/\pi^2)\cdot(\pi^4/90-(1+1/16+\ldots+1/m_0^4))^{1/2} \tag{5.4}$$

Let n be an arbitrary, but fixed odd integer

$$n=\ldots, -5, -3, -1, 1, 3, 5, \ldots$$

and

$$x_0(t)=(n\pi)\varphi_1(t)+0\cdot\varphi_2(t)+\ldots+0\cdot\varphi_{m_0}(t) \tag{5.5}$$

By this definition we see that $f(t, x_0(t))=p(t)$ and so let

$$b=\sqrt{\sum_{j=m_0+1}^{\infty}(p_j/\lambda_j)^2}\leqq k\cdot\|p\| \tag{5.6}$$

$$b'=k'\cdot\|p\|. \tag{5.7}$$

From (4.1)(4.2)(4.3) we have

$$u_i(\gamma)=-\lambda_i\gamma_i+\int_0^1 L\sin(\gamma_1\varphi_1(t)+\ldots+\gamma_{m_0}\varphi_{m_0}(t))\varphi_i(t)\,dt+p_i$$

$$i=1, \ldots, m_0$$

and $\eta=\sqrt{p_1^2+\ldots+p_{m_0}^2}$. From (3.8) it follows that $B=\sqrt{2}$. The matrix $\mathcal{A}$ is diagonal matrix

$$\mathcal{A}=\operatorname{diag}(-L, -\pi^2-L, \ldots, -(m_0-1)^2\pi^2-L) \tag{5.9}$$

and so we have $\sigma=L$.

For $M(c)$ from (4.6) we obtain the bound $M(c)\leqq 2\cdot\sqrt{2}\cdot L$ for $(\gamma_1-n\pi)^2+(\gamma_2)^2+\ldots+(\gamma_{m_0})^2\leqslant c^2$.

All these relations are valid for arbitrary $n=\ldots, -5, -3\cdot -1, 1, 3, 5, \ldots$

The assumption (p6) is valid for

$$L < (m_0 \pi)^2/2 \tag{5.10}$$

Finally from (p8) it follows that the inequality

$$4 \cdot \sqrt{2} \cdot m_0^{3/2} < \frac{L(1-2kL)^2}{(1-kL)((1-kL)\eta + Lb)} \tag{5.11}$$

must be fulfilled.

Theorem 5.1.

The nonlinear boundary value problem (5.1) (5.2) has at least countable many solutions if for the function $p(t)$ and constant L are the conditions (5.10) and (5.11) fulfilled.

Acknowledgement

The author express grateful thanks to prof. V. Šeda, Dr. Sc for initiating and supporting the work and for helpful discussions.

REFERENCES

[1] Birkhoff, G.—Rotta, G. C.: Ordinary differential equations. Ginn and Comp. 1962

[2] Cesari, L.: Functional Analysis and Galerkin's Method. Michigan Math. J., 11 (1964) pp. 386—414

[3] Cesari, L.: Alternative methods in nonlinear analysis. In International Conference on Diff. Eq. (ed. H. Antosiewicz) Academic Press, 1975, pp. 95—148

[4] Ortega, J. M.—Rheinboldt, W. C.: Iterative solutions of nonlinear equations in several variables, Academic Press N. Y. and London 1970. (in Russian Izd. Mir, Moskva 1975 pp. 147—178)

[5] Schneider, Z.: On Fučík's s boundary value problem $x''(t) + x(t) = \alpha(x(t))^- + p(t)$, $x(0) = x(\pi) = 0$ (in print)

[6] Tricomi, F. G.: Integral equations. Interscience publishers Inc. New York 1957 (in Russian Izd. inostr. lit. Moskva 1960)

[7] Vulich, B. Z.: Kratkij kurs teorii funkcij vešč. peremennoj, Nauka, Moskva 1973.

Author's address:

Received: 2. 10. 1981

Zdenek Schneider
Katedra numerickej matematiky MFF UK
Mlynská dolina,
842 15 Bratislava

SÚHRN

GALERKINOVSKÁ APROXIMÁCIA A EXISTENČNÉ VETY PRE ISTÚ NELINEÁRNU OKRAJOVÚ ÚLOHU PRE OBYČAJNÉ DIFERENCIÁLNE ROVNICE

Z. Schneider, Bratislava

V článku sú uvedené postačujúce podmienky pre existenciu riešenia $y(t)$ uvedenej nelineárnej okrajovej úlohy, pre existenciu galerkinovskej aproximácie $x_G(t)$ tohto riešenia. Je uvedený spôsob odhadu chyby rozdielu $y(t) - x_0(t)$ pre danú funkciu $x_0(t)$.

РЕЗЮМЕ

ТЕОРЕМЫ СУЩЕСТВОВАНИЯ И МЕТОД ГАЛЕРКИНА ДЛЯ НЕЛИНЕЙНЫХ КРАЕВЫХ ЗАДАЧ ДЛЯ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

З. Шнайдер, Братислава

Для задачи

$$x''(t) + f(t, x(t)) = 0$$
$$A_0x(0) + A_1x'(0) = 0$$
$$B_0x(l) + B_1x'(l) = 0.$$

показаны достаточные условия для существования решения этой задачи, для существования аппроксимации типа Галеркина. Показан способ оценивания ошибки.