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AN OBSTACLE PROBLEM FOR A CYLINDRICAL SHELL

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We shall be dealing with the problem of the existence and the uniqueness of a solution of a variational inequality for a cylindrical shell with an obstacle. We assume that the bending of the shell is limited by the stiff obstacle. It menas that the function of the bending w is greater or equaled to the prescribed function g. The problem is approximated by the finite element method with the Argyris triangular elements.

1. The existence and uniqueness of a solution

The circular cylindrical shell is an elastic body which occupies the domain:

$$G=\{R\in E^3 \mid R=r(x,\varphi)+zn(\varphi),\ -\frac{1}{2}e(x,\varphi)\leq z\leq\frac{1}{2}e(x,\varphi),\ (x,\varphi)\in\Omega\}$$

where $0<e_0\leq e(x,\varphi)\leq e_1$, $(x,\varphi)\in\Omega$, $n(\varphi)$ is the outward normal to the middle surface and the middle surface is a part of a circular cylinder given by the equations

$$R=r(x,\varphi):\ x_1=x,\ x_2=a\cos\varphi,\ x_3=a\sin\varphi,\ a>0,\ (x,\varphi)\in\Omega$$

where

$$\Omega=\langle -\gamma,\gamma\rangle\times\langle\alpha,\beta\rangle \tag{1.1}$$

We assume for simplicity that the shell is clamped at the edge $\partial\Omega$ what implies that the displacements of the shell u, v, w satisfy the homogeneous boundary conditions.

$$u=v=w=\frac{\partial w}{\partial \nu}=0 \text{ on } \partial\Omega \tag{1.2}$$

where ν is the ontward normal to the boundary $\partial\Omega$. The vertical displacement w

satisfies the unilateral condition with respect to the obstacle

$$w(x, \varphi) \geqq g(x, \varphi), \forall (x, \varphi) \in \Omega \tag{1.3}$$

where the obstacle function g satisfies the conditions

$$g \in C(\bar{\Omega}), \quad g/\partial\Omega \leqslant 0 \tag{1.4}$$

Let us denote

$$H_0^1(\Omega) = \{v \in H^1(\Omega)/v = 0 \text{ on } \partial\Omega\}$$

$$H_0^2(\Omega) = \left\{w \in H^2(\Omega)/w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}$$

where $H^1(\Omega)$, $H^2(\Omega)$ are Sobolev spaces. The space of wirtual displacements of the middle surface of the shell is then the space

$$V = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega) \tag{1.5}$$

V is a Hilbert space equipped with a scalar product

$$(u, v)_V = (u, \psi)_1 + (v, \lambda)_1 + (w, \omega)_2 \tag{1.6}$$

$$\text{for } u = (u, v, w), \quad v = (\psi, \lambda, \omega)$$

and a norm

$$\|u\|_V = (u, u)_V^{1/2}, \quad u \in V \tag{1.7}$$

We introduce further the set of admissible displacements

$$K = \{v = (\psi, \lambda, \omega) \in V/\omega \geqslant g \text{ on } \Omega\} \tag{1.8}$$

Lemma 1. The set K is convex and closed in V.

Proof.: The convexity of K can be seen directly from the definition (1.8). Let us now consider such a sequence $u_n \in K$, $n = 1, 2, \ldots$, that $u_n \to u$ strongly in V. If $u = (u, v, w)$, $u_n = (u_n, v_n, w_n)$, then $w_n \to w$ strongly in $H_0^2(\Omega)$. Due to the imbedding theorem for the space $H_0^2(\Omega)$ ([5]) we have $\lim_{n\to\infty} w_n(x, \varphi) = w(x, \varphi)$ for every point $(x, \varphi) \in \Omega$. As $w_n(x, \varphi) \geqslant g(x, \varphi)$ in Ω, we obtain $w(x, \varphi) \geqslant g(x, \varphi)$ in Ω and hence $u \in K$ what concludes the proof.

We introduce the system of six deformation operators $\{N_i(u)\}$, $i = 1, 2, \ldots, 6$

$$N_1(u) = \frac{\partial u}{\partial x}, \quad N_2(u) = \frac{1}{a}\left(\frac{\partial v}{\partial \varphi} - w\right), \quad N_3(u) = \frac{1}{2}\left(\frac{1}{a}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial x}\right),$$

$$N_4(u) = \frac{\partial^2 w}{\partial x^2}, \quad N_5(u) = \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \varphi^2} + w\right), \quad N_6(u) = \frac{1}{2a}\left(2\frac{\partial^2 w}{\partial x\,\partial \varphi} + \frac{\partial v}{\partial x} - \frac{1}{a}\frac{\partial u}{\partial \varphi}\right) \tag{1.9}$$

and the matrix K of the form

$$K=\begin{bmatrix} B & B\sigma & 0 & 0 & 0 & 0 \\ B\sigma & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 2B(1-\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & D\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2D(1-\mu) \end{bmatrix} \qquad (1.10)$$

where

$$B=\frac{Ee}{1-\mu^2},\quad D=\frac{Ee^3}{12(1-\mu^2)},\quad E>0,\ 0<\mu<1$$

We define now the bilinear form

$$a(u, v)=\int_\Omega N(u)KN(v)a \, dx \, d\varphi, \quad u, v \in V \qquad (1.11)$$

$$N(u)=(N_1(u), \ldots, N_6(u))$$

and the linear functional

$$l(u)=\int_\Omega \left[p_x\left(u-\frac{e}{2}\frac{\partial w}{\partial x}\right)+p_y\left(v-\frac{e}{2a}\frac{\partial w}{\partial \varphi}\right)+p_2 w\right]a \, dx \, d\varphi, \quad u \in V \qquad (1.12)$$

Lemma 1.2. The bilinear form $a(u, v)$ is on the space V bounded and coercive i. e.:

$$|a(u, v)| \leq c_0 \|u\|_V \|v\|_V, \ \forall u, v \in V \qquad (1.13)$$

$$a(u, u) \geq c_1 \|u\|_V^2, \ c_1>0, \ \forall u \in V \qquad (1.14)$$

Proof: We obtain the boundedness (1.13) affer using the Schwarz inequality. Coercivity (1.14) is verified for instance in ([6], chapt. 10.4.5).

We formulate now

Problem P. To find such an element (vector-function) $u \in K$ that

$$a(u, v-u) \geq l(v-u), \ \forall v \in K \qquad (1.15)$$

where the set K is defined in (1.8) and the forms a (.,.), $l(.)$ in (1.11), (1.12)

Theorem 1.1. There exists a unique solution $u \in K$ of the Problem P.

Proof: A vector-function $u \in K$ is a solution of (1.15) if

$$J(u)=\min_{v \in K} J(v), \qquad (1.16)$$

where

$$J(v)=\frac{1}{2} a(v, v)-l(v), \quad v \in K \qquad (1.17)$$

A functional $J(.)$ is due to lemma 1.2 weakly lower semicontinuouns and the set K as convex and closed is weakly closed ([1]). Then there exists due to [1] a function u which fulfils (1.16) and hence the inequality (1.15) holds.

Assume that there exists $u^* \in K$, $u^* \neq u$, which is also a solution of the inequality (1.15). Then we have

$$a(u, u^* - u) \geq l(u^* - u),$$

$$a(u^*, u - u^*) \geq l(u - u^*)$$

and after adding $a(u - u^*, u - u^*) \leq 0$, what is possible only if $u = u^*$.

Hence there exists a unique solution $u \in K$ of the inequality (1.15).

2. Finite elements approximation

The approximation of the Problem P consists of two steps.

1. Substituing the Problem P by the finite dimensional Problem P_h.
2. Numerical solution of the Problem P_h.

The Problem P_h means in this case the finite element approximation of the Problem P. Let $\{\tau_h\}$, $0 \leq h \leq h_0 < \infty$ be the regular system of triangulations of the region Ω. That means that

1. $\bar{\Omega} = \bigcup_{T_i \in \tau_h} T_i, \; i = 1, 2, \ldots, n(h)$

where $h > 0$ is the maximal length of sides of all triangles from τ_h

2. There exists $\vartheta_0 > 0$ such that

$$\min_{\vartheta^h \in \tau_h} \vartheta^h \geq \vartheta_0, \; \forall h \in (0, h_0\rangle$$

where ϑ^h is an arbitrary interior angle in an arbitrary triangle of the triangulation τ_h.

If M_h denotes the set of all nodes of the triangulation τ_h, then we assume

$$M_{h_1} \subset M_{h_2}, \text{ if } h_1 > h_2 \tag{2.1}$$

Let $T \in \tau_h$ be the triangle with vertices a_1, a_2, a_3, mid-points $b_j = \frac{1}{2}(a_{j-1} + a_{j+1})$ and c_j the intersections of the sides a_{j-1}, a_{j+1} and their normales v_j, $j = 1, 2, 3$. (We denote $a_0 = a_3$, $a_4 = a_1$). (See *fig 1*).

It is known from the interpolation theory ([2]) that following 21 values — degrees of freedom

$$\Sigma_T = \{p(a_i),\; Dp(a_i)(a_{i-1} - a_i),\; Dp(a_i)(a_{i+1} - a_i), \tag{2.2}$$

$$1 \leq i \leq 3;\; D^2p(a_i)(a_{j+1} - a_j)^2,\; 1 \leq i, j \leq 3;$$

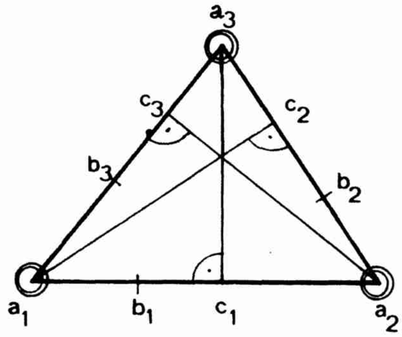

Fig. 1

$$Dp(b_i)(a_i - c_i),\ 1 \leq i \leq 3\}$$

uniquely determine the polynomial of fifth degree $p_5 \in P_5(T_i)$ — Argyris element.

We introduce now for every triangulation τ_h finite dimensional spaces

$$X_h = \{v_h \in C^1(\bar{\Omega})/v_h/T_i \in P_5(T_i),\ \forall T_i \in \tau_h,\ v_h = 0 \text{ on } \partial\Omega\} \quad (2.3)$$

$$Y_h = \{v_h \in C^1(\bar{\Omega})/v_h/T_i \in P_5(T_i),\ \forall T_i \in \tau_h,\ v_h = \frac{\partial v_h}{\partial \nu} = 0 \text{ on } \partial\Omega\} \quad (2.4)$$

The space

$$V_h = X_h \times X_h \times Y_h \quad (2.5)$$

is a finite dimensional subspace of the space V. We define now a finite dimensional approximation of the convex set K by

$$K_h = \{\mathbf{u}_h = (u_h,\ v_h,\ w_h) \in V_h / w_h(a_i) \geq g(a_i),\ \forall a_i \in M_h\} \quad (2.6)$$

We can now proceede to the finite dimensional approximation of the Problem P:

Problem $\mathbf{P_h}$. To find such a vector-function $\mathbf{u}_h \in K_h$, that

$$a(\mathbf{u}_h,\ \mathbf{v}_h - \mathbf{u}_h) \geqq l(\mathbf{v}_h - \mathbf{u}_h),\ \forall \mathbf{v}_h \in K_h \quad (2.7)$$

It can be verified in the same way as in the case of the set $K \subset V$, that the set $K_h \subset V_h$ is convex and closed. Then we obtain the theorem analogous to the Theorem 1.1.

Theorem 2.1. There exists for every $h \in (0, h_0)$ a unique solution $\mathbf{u}_h \in K_h$ of the Problem $\mathrm{P_h}$.

We further show that a sequence $\mathbf{u}_h$ of solutions of the Problems $\mathrm{P_h}$ converges to the solution u of the problem P. We verify at first a weak convergence.

Lemma 2.1. Let $u_h \in K_h$ be a solution of the Problem P_h for every $h \in (0, h_0\rangle$, $u \in K$ be a solution of the Problem P. Then

$$u_h \rightharpoonup u \quad \text{in } V \text{ (weakly)} \tag{2.8}$$

Proof. Choose such a sequence $v_h \in K_h$ that $\|v_h\|_V \leq C_1$ for every $h \in (0, h_0\rangle$. We obtain then due to coercivity of a form $a(.,.)$ the boundedness of the sequence $u_h \in K_h$ of solutions of Problems P_n

$$\|u_h\|_V \leq C, \forall h \in (0, h_0\rangle \tag{2.9}$$

Then there exists such a subsequence chosen from u_h (denoted again by u_h) that

$$u_h \rightharpoonup u^* \quad \text{in } V \text{ (weakly)} \tag{2.10}$$

We have to verify that $u^* = u$ is a solution of the Problem P. We show at first that $u^* = (u^*, v^*, w^*) \in K$ i. e.

$$w^*(x, \varphi) \geq g(x, \varphi), \forall (x, \varphi) \in \Omega \tag{2.11}$$

As $u_h = (u_h, v_h, w_h) \in K_h$, we have

$$w_h(a_i) \geq g(a_i), \forall a_i \in M_h \tag{2.12}$$

Let $\varepsilon > 0$. As the functions w_h, g are uniformly continuons on $\bar{\Omega}$, there exists such a number $h_1 \in (0, h_0\rangle$ that

$$w_h(x, \varphi) \geq g(x, \varphi) - \varepsilon, \forall (x, \varphi) \in \Omega \tag{2.13}$$

The set $K_\varepsilon = \{u = (u, v, w) \in V / w(x, \varphi) \geq g(x, \varphi) - \varepsilon, \forall (x, \varphi) \in \Omega\}$ is convex, closed and hence weakly closed in V. Then we have with respect to (2.10) $u^* \in K_\varepsilon$ and hence

$$w^*(x, \varphi) \geq g(x, \varphi) - \varepsilon, \forall (x, \varphi) \in \Omega \tag{2.14}$$

As $\varepsilon > 0$ is an arbitrary positive number, we obtain

$$w^*(x, \varphi) \geq g(x, \varphi), \forall (x, \varphi) \in \Omega \tag{2.15}$$

what means $u^* \in K$.

It remains us to show that u^* is a solution of the Problem P. Let $v \in K \cap [C^\infty(\bar{\Omega})]^3$. Denote v_h^I the Hermit interpolation polynomical belonging to the function $v \in K$. Obviously $v_h^I \in K_h$, because $v_h^I(a_i) = v(a_i) \geq g(a_i)$ for every $a_i \in M_h$. That means

$$a(u_h, v_h^I - u_h) \geq l(v_h^I - u_h) \tag{2.16}$$

Using the estimate ([2])

$$\|v - v_h^I\|_V \leq ch \|v\|_V, \ \forall v \in V \cap [C^\infty(\bar{\Omega})]^3 \quad (2.17)$$

we obtain from (2.16) after limiting

$$a(u^*, v - u^*) \geq l(v - u^*), \ \forall v \in K \cap [C^\infty(\bar{\Omega})]^3 \quad (2.18)$$

It is verified in ([3], chapt. 3.3.2) that the set $K_W \cap [C^\infty(\bar{\Omega})]$ is dense in K_W, where

$$K_W = \{ w \in H_0^2(\Omega) / w(x, \varphi) \geq g(x, \varphi), \ \forall (x, \varphi) \in \Omega \}$$

Then the set $K \cap [C^\infty(\bar{\Omega})]^3$ is dense in K and the inequality (2.18) holds also for arbitrary $v \in K$. Hence u^* is a solution for the Problem P. We have then $u^* = u$ due to the unicity of a solution of P and the Lemma is verified.

The following theorem expresses the strong convergence of the sequence u_h.

Theorem 2.2. Let $u_h \in K_h$, $h \in (0, h_0\rangle$ and $u \in K$ are solutions of the Problems P_h and P respectively. Then

$$\lim_{h \to 0^+} \|u_h - u\|_V = 0 \quad (2.19)$$

Proof.: Using the inequalities (1.14), (2.7) we obtain the estimates

$$c_1 \|u_h - u\|_V^2 \leq a(u_h - u, u_h - u) \leq a(u, u - u_h) + a(u_h, v_h^I - u) - l(v_h^I - u_h),$$
$$c_1 > 0; \ \forall v \in K \cap [C^\infty(\bar{\Omega})]^3$$

After limiting we arrive at

$$0 \leq c_1 \overline{\lim_{h \to 0^+}} \|u_h - u\|_V^2 \leq a(u, v - u) - l(v - u), \quad (2.20)$$
$$\forall v \in K \cap [C^\infty(\bar{\Omega})]^3$$

As the set $K \cap [C^\infty(\bar{\Omega})]^3$ is dense in K the inequality (2.20) holds for every $v \in K$. We can now pat $v = u$ and arrive at

$$0 \leq \underline{\lim_{h \to 0^+}} \|u_h - u\|_V^2 \leq \overline{\lim_{h \to 0^+}} \|u_h - u\|_V^2 \leq 0$$

and the relation (2.19) follows immediately.

Some of the methods of quadratic programming can be applied to the solving the Problem P_h. We shall use the same approach as in [4] in the case of plate. Let $\varphi_1, \ldots, \varphi_{n_h}$ and $\psi_1, \ldots, \psi_{m_h}$ be the interpolating basis of X_h and ψ_h. Then the system of vector-functions $\Phi_1, \ldots, \Phi_{m_h + 2n_h}$ defined by

$$\Phi_j = \begin{array}{ll} (0, 0, \psi_j), & j = 1, \ldots, m_h \\ (0, \varphi_{j - m_h}, 0), & j = m_h + 1, \ldots, m_h + n_h \\ (\varphi_{j - m_h - n_h}, 0, 0), & j = m_h + n_h + 1, \ldots, m_h + 2n_h \end{array} \quad (2.21)$$

is the basis of $V_h = X_h \times X_h \times Y_h$. Let the first L_h functions correspond to the values of the basis functions $\{\psi_j\}$ in the inner nodes of the triangulation τ_h. Then every vector-function $v=(\psi, \lambda, \omega) \in V_h$ can be expressed in the form

$$v=\sum_{j=1}^{P_h} q_j \Phi_j, \quad P_h = m_h + 2n_h \tag{2.22}$$

where

$$q_j = \omega(a_j), \; a_j \in M_h^0, \; j=1, \ldots, L_h \tag{2.23}$$

M_h^0 is the set of all interval nodes of triangulation τ_h. The definition of the convex set K_h implies

$$v \in K_h \Leftrightarrow q=(q_1, \ldots, q_{L_h})^T \in K_{L_h} \tag{2.24}$$

where

$$K_{L_h} = \{ q_h \in E_{L_h} / q_j \geqslant g(a_j), \; a_j \in M_h^0, \; j=1, \ldots, L_h \} \tag{2.25}$$

Substituing (2.22) into (2.7) and using a variational formulation, we obtain the equivalent problem to the Problem $\mathrm{P_h}$

Problem $\mathrm{P_h^*}$: To find such $q^* \in K_{L_h}$ that

$$L(q_h^*) = \min_{q \in K_{L_h}} L(q_h)$$

where

$$L(q_h) = J(v_h) = \frac{1}{2} q^T A q - l^T q,$$

$$A=(a_{ij})_{i,j=1}^{P_h}, \; a_{ij} = a(\Phi_i, \Phi_j), \; l_j = l(\Phi_j),$$

$$i, j=1, \ldots, P_h$$

One of the effective numerical method for solving the Problem $\mathrm{P_h^*}$ is the modification of SOR method [3]:

Let $q^0 \in K_{L_h}$ be the starting element,

$$q_i^{m+1/2} = -\frac{1}{a_{ii}} \left(\sum_{j=1}^{i-1} a_{ij} q_j^{m+1} + \sum_{j=i+1}^{P_h} a_{ij} q_j^m - l_i \right)$$

$$q_i^{m+1} = \max \{ q(a_i), (1-\omega) q_i^m + \omega q_i^{m+1/2} \}, \; i=1, \ldots, L_h$$

$$q_i^{m+1} = (1-\omega) q_i^m + \omega q_i^{m+1/2}, \; i=L_h+1, \ldots, P_h; \; m=1, 2, \ldots$$

where $\omega \in (0, 2)$ is a chosen weighting factor.

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SÚHRN

ÚLOHA S PREKÁŽKOU PRE VALCOVÚ ŠKRUPINU

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V práci je skúmaná jednostranná úloha pre valcovú škrupinu. Dokázaná je existencia riešenia, jednoznačnosť riešenia a použiteľnosť približnej metódy.

РЕЗЮМЕ

ЗАДАЧА С ПРЕПЯТСТВИЕМ ДЛЯ ЦИЛИНДРИЧЕСКОЙ ОБОЛОЧКИ

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Рассматривается односторонная задача для цилиндрической оболочки. Доказано существование решения, единственность решения и применимость приблизительного метода.