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EXISTENCE AND BOUNDEDNESS OF SOLUTIONS OF THE PERTURBED NONLINEAR DIFFERENTIAL EQUATION WITH DELAY

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In [1] the author has obtained some existence theorems for the initial value problem (IVP)

$$\frac{dy}{dt}=f(t, y(t))+g(t, y(t), y[h_1(t)], \ldots, y[h_m(t)]) \tag{1}$$

$$\begin{gathered} y(t_0^+)=p(t_0)=y_0 \\ y[h_i(t)]\equiv p[h_i(t)], \quad h_i(t)\leqq t_0, \; i=1, \ldots, m, \end{gathered} \tag{3}$$

where the unperturbed system is nonlinear

$$\frac{dx}{dt}=f(t, x) \tag{2}$$

and each of x, y is n-dimensional column vector, by using the nonlinear variation of constant formula due to Alexejev [2] and a comparison principle.

In the present paper we shall continue this study by obtaining further conditions which ensure the existence of a solution of the IVP (1), (3) and give certain different type of its estimation as well. Here the results of [3] and the Bellman's lemma will be applied.

Let R^n denote the n-dimensional vector space, $|.|$ any appropriate vector norm and $\|.\|$ any compatibile matrix norm.

Assumptions (a)—(d) given below are valid throughout this paper and will not be repeated in formulations of particular theorems.

Suppose that:

(a) $f(t, y), \dfrac{\partial f(t, y)}{\partial y}\in C[I\times R^n, R^n]$, where $I=[a, \infty)$;

(b) $g(t, v, u_1, \ldots, u_m)\in C[I\times R^{n(m+1)}, R^n]$;

(c) $h_i(t) \in C[I, R]$ such that $h_i(t) \leqq t$ for all $t \in I$, $i=1, \ldots, m$;

(d) $p(t) \in C[E_{t_0}, R^n]$, $t_0 \in I$, $E_{t_0} = \bigcup_{i=1}^{m} E_{t_0}^i$ is the initial set, where $E_{t_0}^i = \left(\inf_{t \in J} h_i(t), t_0\right]$, $i=1, \ldots, m$ and we shall assume that for every $i=1, \ldots, m$ there exists $t_i \in [t_0, \infty) = J$ such that $h_i(t_i) = t_0$. If $\inf_{t \in J} h_i(t) = \min_{t \in J} h_i(t)$ we shall put $E_{t_0}^i = \left[\inf_{t \in J} h_i(t), t_0\right]$.

We shall denote $x(t, t_0, x_0)$ the solution of (2) passing through the point (t_0, x_0). It will be always assumed that $x(t, t_0, x_0)$ has the following property:

(e) for every $x_0 \in R^n$, $x(t, t_0, x_0)$ as a function of t can be continued to J.

Remark. The meaning of some of these conditions had been explained in [4], pp. 131—132 and [5].

Theorem 1. Let us assume that

i) there exists a function $k(t, s, r) \in C[L = \{(t, s)\colon a \leqq s < t < \infty\} \times R_+, R_+]$ nondecreasing in the last variable such that for the matrix $F(t, t_0, x_0) = \dfrac{\partial x(t, t_0, x_0)}{\partial x_0}$ the inequality

$$\|F(t, t_0, x_0)\| \leqq k(t, t_0, |x_0|), \ (t, t_0) \in L, \ x_0 \in R^n \tag{4}$$

holds;

ii) for each $(s, v, u_1, \ldots, u_m) \in J \times R^{n(m+1)}$ the relation

$$|g(s, v, u_1, \ldots, u_m)| < n(s, |v|, |u_1|, \ldots, |u_m|) \tag{5}$$

holds, where $n \in C[J \times R_+^{m+1}, R_+]$ is nondecreasing in all variables with the exception of the first one;

iii) there exists a function $r(t) \in C[E_{t_0} \cup J, R_+]$ such that $|p(t)| < r(t)$, $t \in E_{t_0}$, $|x(t)| \leqq r(t)$, $t \in J$ where $x(t)$ is a solution of (2) satisfying the initial condition $x(t_0) = x_0 = p(t_0)$;

iv) there exists a maximal solution $Z(t)$ of the integral equation

$$\begin{aligned} &z(t) = r(t), \ t \in E_{t_0} \\ &z(t) = r(t) + \int_{t_0}^{t} k(t, s, z(s)) n(s, z(s), z[h_1(s)], \ldots, z[h_m(s)]) \, ds, \end{aligned} \tag{6}$$

$t \in J$ which is bounded on J.

Then the IVP (1), (3) has at least one solution $y(t)$ on J. Moreover, this solution satisfies the inequality

$$|y(t)| < M \text{ on } J, \tag{7}$$

where the constant M is determined by (8).

Proof. With regard to the assumptions there exists a positive constant M such that for $t \in J$ the inequality

$$Z(t) \leqq M \tag{8}$$

holds. For $y(t)$, in accordance with the assumptions,

$$|y(t_0)| < r(t_0) = Z(t_0) \tag{9}$$

is true, so as the inequality (7) is valid at least at a right neighbourhood of the point t_0. We will show that $y(t)$ exists for all $t \in J$ and $|y(t)| < M$. If this were false, with respect to the Lemma of [1] there exists the first point t_1 on the right from t_0 such that $|y(t_1)| = M$. It is known [2] that the solution $y(t)$ of the IVP (1), (3) also fulfils the system

$$\begin{aligned} &y(t) = p(t), \quad t \in E_{t_0} \\ &y(t) = x(t) + \int_{t_0}^{t} F(t, s, y(s)) \;\; g(s, y(s), y[h_1(s)], \ldots, y[h_m(s)]) \, ds, \end{aligned} \tag{10}$$

$t \in J$, where $x(t)$ is the solution of (2) determined by the initial condition $x(t_0) = p(t_0)$, wherefrom with respect to the (4) and (5) we have

$$\begin{aligned} &|y(t)| = |p(t)|, \quad t \in E_{t_0} \\ &|y(t)| < |x(t)| + \int_{t_0}^{t} k(t, s, |y(s)|) \;\; n(s, |y(s)|, |y[h_1(s)]|, \ldots \\ &\ldots, |y[h_m(s)]|) \, ds, \quad t_0 \leqq t \leqq t_1. \end{aligned}$$

Since $|p(t)| < r(t)$, $t \in E_{t_0}$, using the theorem 3 of [3] one obtains $|y(t)| < Z(t) \leqq M$ for each considered t. Thus the point $t_1 > t_0$ such that $|y(t_1)| = M$ cannot exist. This contradiction proves that $y(t)$ can be extended to the whole interval J keeping the inequality (7). The proof is complete.

Theorem 2. Let us assume that

i) there exists a function $K(t, s, r, p_1, \ldots, p_m) \in C[L \times R_+^{m+1}, R_+]$ nondecreasing in all variables with the exception of the first two ones such that

$$|F(t, s, v) g(s, v, u_1, \ldots, u_m)| < K(t, s, r, p_1, \ldots, p_m); \tag{11}$$

ii) there exists a function $r(t) \in C[E_{t_0} \cup J, R_+]$ such that $|p(t)| < r(t)$, $t \in E_{t_0}$, $|x(t)| \leqq r(t)$, $t \in J$ where $x(t)$ is a solution of (2) satisfying the initial condition $x(t_0) = x_0 = p(t_0)$;

iii) there exists a total solution $z^0(t)$ of the integral equation

$$\begin{aligned} &z(t) = r(t), \quad t \in E_{t_0} \\ &z(t) = r(t) + \int_{t_0}^{t} K(t, s, z(s), z[h_1(s)], \ldots, z[h_m(s)]) \, ds, \quad t \in J \end{aligned} \tag{12}$$

on J.

Then the IVP (1), (3) has at least one solution $y(t)$ on J. Moreover, this solution satisfies the inequality

$$|y(t)| < z^0(t) \text{ on } J. \tag{13}$$

Proof. We suppose the maximal interval of existence of a solution $y(t)$ of the IVP (1), (3) is $[t_0, T]$, $T < \infty$. For $t \in [t_0, T]$ the $z^0(t)$ is a bounded function. With regard to the assumptions the following relations

$$|y(t)| \leqq |p(t)|, \quad t \in E_{t_0}$$

$$|y(t)| < |x(t)| + \int_{t_0}^{t} K(t, s, |y(s)|, |y[h_1(s)]|, \ldots, |y[h_m(s)]|) \, ds \leqq$$

$$\leqq r(t) + \int_{t_0}^{t} K(t, s, |(s)|, |y[h_1(s)]|, \ldots, |y[h_m(s)]|) \, ds, \quad t_0 \leqq t < T$$

are true, wherefrom using the Theorem 4 of [4] one obtains that for $t \in [t_0, T)$ the inequality (13) holds. It follows from this, however, that $y(t)$ is a bounded function on the maximal interval of its existence. Therefore, in accordance with the Lemma of [1] there must be $T = \infty$ which completes the proof.

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SÚHRN

EXISTENCIA A OHRANIČENOSŤ RIEŠENÍ PERTURBOVANEJ NELINEÁRNEJ DIFERENCIÁLNEJ ROVNICE S ONESKORENÍM

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V práci sa vyšetruje nelineárna perturbovaná diferenciálna rovnica s oneskorením, pričom odpovedajúca naperturbovaná rovnica je tiež nelineárna. Využitím istých integrálnych nerovností a Bellmanovej lemy sú stanovené rôzne podmienky pre existenciu riešení počiatočnej úlohy a ich odhady.

РЕЗЮМЕ

СУЩЕСТВОВАНИЕ И ОГРАНИЧЕННОСТЬ РЕШЕНИЙ ВОЗМУЩЕННОГО НЕЛИНЕЙНОГО ФИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ЗАПАЗДЫВАНИЕМ

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В статье рассматривается нелинейное возмущенное дифференциальное уравнение с запаздывающим аргументом, где невозмущенное уравнение тоже нелинейное. Через простредничество каких-то интегральных неравенств и при воспользовании леммы Беллмана установлены условия существования решений начальной задачи этого возмущенного уравнения и их оценки.